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A Remark on Pillen's Theorem for Projective Indecomposable $kG(n)$-Modules

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Abstract. Let g be a connected, semisimple and simply connected algebraic group defined and split over the finite field of order p, and let $g(n)$ be the corresponding finite Chevalley group and g_n the n-th Frobenius kernel. Pillen has proved that for a $3(h-1)$-deep and p^n-restricted weight λ, the G-module $Q_n(\lambda)$ which is extended from the G_n-PIM for λ has the same socle series as the corresponding $kG(n)$-PIM $U_n(\lambda)$. Here we remark that this fact already holds for λ being $2(h-1)$-deep.
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1. Introduction

Let G be a connected, semisimple and simply connected algebraic group over an algebraically closed field k of characteristic $p > 0$ which is defined and split over $\mathbb{F}_p$. Let $F : G \to G$ be the n-th Frobenius map, and let G_n be the (scheme-theoretic) kernel of F, which is called the n-th Frobenius kernel. The finite subgroup consisting of all fixed points of F on G is denoted by $G(n)$ and called a finite Chevalley group.

In order to study the projective indecomposable $kG(n)$-modules $U_n(\lambda)$, it is effective to consider the corresponding projective indecomposable G_n-modules $Q_n(\lambda)$. When p is not "too small", any $Q_n(\lambda)$ can be extended to a certain G-module and is a direct sum of some U_n's as a $kG(n)$-module. The multiplicities of U_n's in $Q_n(\lambda)$ are completely determined by Jantzen and Chastkofsky (see Proposition 2.2). Moreover, $Q_n(\lambda)$ also plays an important role when we study the Loewy series of $U_n(\lambda)$. Indeed, when $G = \mathrm{SL}(2,k)$, Andersen, Jørgensen and Landrock [2, Theorem 4.3] have proved that the Loewy series of the $kG(n)$-PIM $U_n(\lambda)$ is obtained from that of the G-module $Q_n(\lambda)$ if $p \nmid \lambda + 1$. When G is arbitrary, the situation gets more complicated. However, Pillen has proved that a similar fact holds if λ is $3(h-1)$-deep [9, Theorem 3.3], where h is the Coxeter number of G.

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In this paper, we extend this result of Pillen's to the case of $2(h-1)$-deep weight λ:

Theorem 1.1. *Suppose that a p^n-restricted weight λ is $2(h-1)$-deep. Then we have* $\mathrm{soc}^i_G Q_n(\lambda) = \mathrm{soc}^i_{G(n)} U_n(\lambda)$ *and* $\mathrm{rad}^i_G Q_n(\lambda) = \mathrm{rad}^i_{G(n)} U_n(\lambda)$ *for each i.*

Actually, most of the results written in [9, §3] can be extended to the $2(h-1)$-deep case as the theorem. These are argued in the section 3. The method of the proof is essentially similar to Pillen's, but we need a little modification because in this $2(h-1)$-deep case the various modules $Q_n(\mu)$ which appear in the proof are not always indecomposable as $kG(n)$-modules. The key point is that the formula on the multiplicity of $U_n(\mu)$ in $Q_n(\lambda)$ with $\mu \neq \lambda$ limits severely the weight μ.

2. Preliminaries

Let G be the one as in the introduction, but for simplicity we assume that G is (almost) simple for the rest of the paper. The results can be easily extended to the semisimple case. Let T be a maximal split torus of G. Let $X = X(T)$ be the character group, Φ the root system relative to G and T, and Φ^+ the set of positive roots. Let Δ be the set of simple roots. Let α_0 be the highest short root. On the euclidean space $\mathbb{E} = X \otimes \mathbb{R}$ we can define an inner product $\langle \cdot, \cdot \rangle$ which is invariant under the action of the Weyl group $W = N_G(T)/T$. For each root $\alpha \in \Phi$ we set $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ and call it the coroot of α. Let ρ be half the sum of all positive roots, and set $h = \langle \rho, \alpha_0^\vee \rangle + 1$, which is called the Coxeter number. The elements of the subsets

$$X^+ = \{\lambda \in X | \langle \lambda, \alpha^\vee \rangle \geq 0, \forall \alpha \in \Delta\}$$

and

$$X_n = \{\lambda \in X^+ | \langle \lambda, \alpha^\vee \rangle < p^n, \forall \alpha \in \Delta\}$$

are called the dominant weights and the p^n-restricted weights respectively. X^+ parametrizes the simple (rational) G-modules, and let $L(\lambda)$ denote the simple G-module of highest weight $\lambda \in X^+$. For $\lambda \in X^+$, we can write $\lambda = \lambda_0 + p^n \lambda_1$ uniquely with $\lambda_0 \in X_n$ and $\lambda_1 \in X^+$, and then we have $L(\lambda) \cong L(\lambda_0) \otimes L(p^n \lambda_1)$ as G-modules (see [7, II 3.16 Proposition]). X_n parametrizes the simple $kG(n)$-modules and they are obtained by restricting the simple G-modules $L(\lambda)$ with $\lambda \in X_n$ to $G(n)$. For $\lambda \in X^+$ let $V(\lambda)$ be the Weyl G-module with highest weight λ.

If V is a G-module, we denote by $\mathrm{wt}(V)$ the set of all distinct weights of V, and denote by $[V : L]_G$ the multiplicity of a simple G-module L in the composition factors of V. For $\lambda \in \mathrm{wt}(V)$ let

$$V_\lambda = \{v \in V | tv = \lambda(t)v, \ \forall t \in T\}$$

be the weight space of λ in V. Set $\mathrm{ch}(V) = \sum_{\lambda \in X} (\dim V_\lambda)\, e(\lambda)$, which is called the formal character of V, where $\{e(\lambda) | \lambda \in X\}$ is the canonical basis of the group ring

$\mathbb{Z}[X]$. We denote by $\mathrm{soc}^i_G V$ (resp. $\mathrm{soc}^i_{G(n)} V$) the i-th G-(resp. $kG(n)$-) socle of a module V, and similarly by $\mathrm{rad}^i_G V$ (resp. $\mathrm{rad}^i_{G(n)} V$) the i-th G-(resp. $kG(n)$-) radical of V. For $\lambda \in X_n$, let $Q_n(\lambda)$ (resp. $U_n(\lambda)$) be the G_n-(resp. $kG(n)$-) projective cover of $L(\lambda)$. It is well known that any projective indecomposable G_n-module $Q_n(\lambda)$ can be uniquely extended to a G-module if $p \geq 2(h-1)$ (see [5, §4]). Then the G-module $Q_n(\lambda)$ is also projective as a $kG(n)$-module and has a summand isomorphic to $U_n(\lambda)$ with multiplicity one.

Let W_p be the affine Weyl group. W_p acts on X as the dot action $w \cdot \lambda = w(\lambda+\rho) - \rho$ for $w \in W_p$ and $\lambda \in X$.

We call that $\lambda \in X$ is a-deep ($a > 0$) if $p | \langle \lambda+\rho, \alpha^\vee \rangle + c$ implies $|c| \geq a$ for all $\alpha \in \Phi^+$, and that a G-module V is a-deep if any simple composition factor of V has an a-deep highest weight. To argue the main results, we use several facts on G-modules.

Proposition 2.1. *Let $\lambda, \nu \in X^+$ with λ lying in an alcove A.*

(1) (cf. [4, Lemma]) *Suppose that $\lambda+\gamma$ lies in the closure of the alcove A for any $\gamma \in \mathrm{wt}(L(\nu))$. Then*

$$L(\lambda) \otimes L(\nu) \cong \bigoplus_{\pi} (\dim L(\nu)_\pi) L(\lambda+\pi)$$

as G-modules, where π runs over all distinct weights of $L(\nu)$ with $\lambda+\pi$ lying in the upper closure of A.

(2) ([8, Lemma 5.1 (2)]) *Suppose that $p \geq 2(h-1)$ and that $\lambda+\gamma$ lies in the alcove A for any $\gamma \in \mathrm{wt}(L(\nu))$. Then*

$$Q_n(\lambda) \otimes L(\nu) \cong \bigoplus_{\pi \in \mathrm{wt}(L(\nu))} (\dim L(\nu)_\pi) Q_n(\lambda+\pi)$$

as G-modules.

Proof. We shall prove only (1). For convenience, we set $V(\lambda) = 0$ when $\lambda \notin X^+$. By Brauer's formula and [7, II Proposition 7.11], we get

$$\begin{aligned} \mathrm{ch}(V(w\cdot\lambda))\ \mathrm{ch}(L(\nu)) &= \sum_{\pi \in \mathrm{wt}(L(\nu))} (\dim L(\nu)_\pi)\ \mathrm{ch}(V(w\cdot\lambda+\pi)) \\ &= \sum_{\pi \in \mathrm{wt}(L(\nu))} (\dim L(\nu)_\pi)\ \mathrm{ch}(T_\lambda^{\lambda+\pi} V(w\cdot\lambda)) \end{aligned}$$

for each $w \in W_p$ with $w \cdot \lambda \in X^+$, where T^μ_λ is the translation functor from λ to μ.

Now we can write $\mathrm{ch}(L(\lambda)) = \sum_{w \in W_p,\ w\cdot\lambda \in X^+} b_{w\cdot\lambda}\ \mathrm{ch}(V(w\cdot\lambda))$, where

$b_{w\cdot\lambda} \in \mathbb{Z}$ and the sum is finite. Then we have

$$\begin{aligned}
\mathrm{ch}(L(\lambda)\otimes L(\nu)) &= \sum_{w\in W_p,\ w\cdot\lambda\in X^+} b_{w\cdot\lambda}\ \mathrm{ch}(V(w\cdot\lambda))\ \mathrm{ch}(L(\nu)) \\
&= \sum_{w\in W_p,\ w\cdot\lambda\in X^+} b_{w\cdot\lambda} \sum_{\pi\in\mathrm{wt}(L(\nu))} (\dim L(\nu)_\pi)\ \mathrm{ch}(T_\lambda^{\lambda+\pi}V(w\cdot\lambda)) \\
&= \sum_{\pi\in\mathrm{wt}(L(\nu))} (\dim L(\nu)_\pi) \sum_{w\in W_p,\ w\cdot\lambda\in X^+} b_{w\cdot\lambda}\ \mathrm{ch}(T_\lambda^{\lambda+\pi}V(w\cdot\lambda)) \\
&= \sum_{\pi\in\mathrm{wt}(L(\nu))} (\dim L(\nu)_\pi)\ \mathrm{ch}(T_\lambda^{\lambda+\pi}L(\lambda)),
\end{aligned}$$

where the last equality follows from the exactness of the translation functors. Now by [7, II Proposition 7.15] $\mathrm{ch}(T_\lambda^{\lambda+\pi}L(\lambda))$ is isomorphic to $L(\lambda+\pi)$ if $\lambda+\pi$ in the upper closure of the alcove A, and otherwise zero. Since the highest weights of any two non-isomorphic composition factors of $L(\lambda)\otimes L(\nu)$ are not linked and since self-extensions of simple G-modules do not exist, the tensor product must be semisimple as a G-module by the linkage principle, and hence the claim follows. ■

The following proposition shows that the multiplicity $(Q_n(\lambda) : U_n(\mu))$ of a $kG(n)$-PIM $U_n(\mu)$ in the restriction (to $G(n)$) of the G-module $Q_n(\lambda)$ is obtained in terms of multiplicities of G-composition factors:

Proposition 2.2. ([6, 2.10 Corollar 2], [3, §3 Corollary 2])
Suppose that $p \geq 2(h-1)$. Then

$$(Q_n(\lambda) : U_n(\mu)) = \sum_{\nu\in X^+} [L(\mu)\otimes L(\nu) : L(\lambda + p^n\nu)]_G$$

for any $\lambda, \mu \in X_n$.

Remark. Using this proposition, Pillen proves that if λ is $(h-1)$-deep, then $Q_n(\lambda) = U_n(\lambda)$ (see [8, Lemma 6.1 (1)]).

3. Main results

The main results are obtained as consequences of the following lemma.

Lemma 3.1. *Let V be a $2(h-1)$-deep G-module. Suppose that V satisfies $\langle \xi, \alpha_0^\vee\rangle \leq 2(p^n-1)(h-1)$ for any $\xi \in \mathrm{wt}(V)$. Then $\mathrm{soc}_G^i V = \mathrm{soc}_{G(n)}^i V$ for each i.*

This is a "refinement" of Pillen's [9, Lemma 3.1]. Though the weight assumption here is a bit stronger than the p^n-boundedness (i.e. $\langle \xi, \alpha_0^\vee\rangle < 2p^n(h-1)$) there, it does not affect the proofs of the results.

Proof. Since any quotient G-module of V also satisfies all of the hypotheses of V, it suffices to prove that $\mathrm{soc}_G V = \mathrm{soc}_{G(n)} V$ because of the induction on i.

Suppose that $\mathrm{soc}_G V \cong \bigoplus_{\lambda\in\mathcal{X}} L(\lambda_0)\otimes L(p^n\lambda_1)$, where $\lambda = \lambda_0 + p^n\lambda_1$ with $\lambda_0 \in X_n$, $\lambda_1 \in X^+$ and $\mathcal{X}$ is a set of dominant weights (but respecting multiplicities). As in the proof of [1, Lemma 2.2] or [9, Lemma 3.1], the embedding

$$\mathrm{soc}_G V \to \bigoplus_{\lambda\in\mathcal{X}} Q_n(\lambda_0)\otimes L(p^n\lambda_1)$$

can be extended to all of V.

The inclusion $\mathrm{soc}_G V \subseteq \mathrm{soc}_{G(n)} V$ follows from the fact that each $L(\lambda) = L(\lambda_0)\otimes L(p^n\lambda_1)$ for $\lambda\in\mathcal{X}$ is semisimple as a $kG(n)$-module. Indeed, $L(p^n\lambda_1)$ is isomorphic to $L(\lambda_1)$ as a $kG(n)$-module and the hypothesis of the weights of V implies $\langle\lambda_1,\alpha^\vee\rangle < 2(h-1)$ for any $\alpha\in\Phi^+$, and then the G-module $L(\lambda_0)\otimes L(\lambda_1)$ is p^n-restricted and semisimple by Proposition 2.1 (1) since λ_0 is $2(h-1)$-deep.

Again by Proposition 2.1 we get

$$\begin{aligned}\mathrm{soc}_G V &\cong \bigoplus_{\lambda\in\mathcal{X}} L(\lambda_0)\otimes L(p^n\lambda_1)\\ &\cong \bigoplus_{\lambda\in\mathcal{X}} L(\lambda_0)\otimes L(\lambda_1)\\ &\cong \bigoplus_{\lambda\in\mathcal{X}}\ \bigoplus_{\nu\in\mathrm{wt}(L(\lambda_1))} (\dim L(\lambda_1)_\nu) L(\lambda_0+\nu)\end{aligned}$$

and

$$\begin{aligned}\bigoplus_{\lambda\in\mathcal{X}} Q_n(\lambda_0)\otimes L(p^n\lambda_1) &\cong \bigoplus_{\lambda\in\mathcal{X}} Q_n(\lambda_0)\otimes L(\lambda_1)\\ &\cong \bigoplus_{\lambda\in\mathcal{X}}\ \bigoplus_{\nu\in\mathrm{wt}(L(\lambda_1))} (\dim L(\lambda_1)_\nu) Q_n(\lambda_0+\nu)\end{aligned}$$

as $kG(n)$-modules. Note that each $L(\lambda_0+\nu)$ is mapped into $Q_n(\lambda_0+\nu)$ under the compositions of these isomorphisms and the embedding

$$\mathrm{soc}_G V \to \bigoplus_{\lambda\in\mathcal{X}} Q_n(\lambda_0)\otimes L(p^n\lambda_1).$$

Suppose that $\mathrm{soc}_G V \neq \mathrm{soc}_{G(n)} V$. Then there exists a $kG(n)$-composition factor $L(\zeta)$ of $\mathrm{soc}_{G(n)} V$ with $\zeta\in X_n$ which is not contained in $\mathrm{soc}_G V$. By Proposition 2.2, each $Q_n(\lambda_0+\nu)$ (considered as a $kG(n)$-module) can be written as

$$Q_n(\lambda_0+\nu) = U_n(\lambda_0+\nu) \oplus (\text{a direct sum of some } U_n(\eta)\text{'s with } \eta\neq\lambda_0+\nu).$$

It follows that there exist $\lambda\in\mathcal{X}$ and $\nu\in X^+$ with $\lambda_0+\nu\neq\zeta$ such that $(Q_n(\lambda_0+\nu) : U_n(\zeta)) \neq 0$. Again by Proposition 2.2, there exists a nonzero weight $\gamma\in X^+$ such that $L(\lambda_0+\nu+p^n\gamma)$ occurs at least once as a G-composition factor of $L(\zeta)\otimes L(\gamma)$. Note that $\zeta-(p^n-1)\gamma \geq \lambda_0+\nu$. The simple $kG(n)$-module $L(\zeta)$ is a $kG(n)$-composition factor of the restriction of a certain G-composition factor $L(\mu) = L(\mu_0)\otimes L(p^n\mu_1)$ of V. Now we claim that $\langle\mu_1+\gamma,\alpha_0^\vee\rangle > 2(h-1)$. Indeed, if $\langle\mu_1+\gamma,\alpha_0^\vee\rangle \leq 2(h-1)$, then the G-module $L(\mu_1)\otimes L(\gamma)$ is semisimple and has the highest weight $\mu_1+\gamma$. Then the $2(h-1)$-deepness of μ_0 and Proposition 2.1

(1) implies that the G-module $L(\mu_0)\otimes L(\mu_1)\otimes L(\gamma)$ (hence $L(\zeta)\otimes L(\gamma)$) must be p^n-restricted, and hence $L(\lambda_0+\nu+p^n\gamma)$ does not appear in the composition factors of $L(\zeta)\otimes L(\gamma)$, which is contradiction. Now we have

$$\begin{aligned}\langle\mu,\alpha_0^\vee\rangle &= \langle\mu_0+p^n\mu_1,\alpha_0^\vee\rangle\\ &= \langle\mu_0+\mu_1,\alpha_0^\vee\rangle+(p^n-1)\langle\mu_1,\alpha_0^\vee\rangle\\ &\geq \langle\zeta,\alpha_0^\vee\rangle+(p^n-1)\langle\mu_1+\gamma,\alpha_0^\vee\rangle-(p^n-1)\langle\gamma,\alpha_0^\vee\rangle\\ &= \langle\zeta-(p^n-1)\gamma,\alpha_0^\vee\rangle+(p^n-1)\langle\mu_1+\gamma,\alpha_0^\vee\rangle\\ &> \langle\lambda_0+\nu,\alpha_0^\vee\rangle+2(p^n-1)(h-1)\\ &\geq 2(p^n-1)(h-1),\end{aligned}$$

which contradicts the weight hypothesis of V. ■

Remarks. (1) We can easily check that this lemma holds for not only Chevalley groups but also twisted groups, because there is a twisted analogue to Proposition 2.2 (see [6, 2.10 Corollar 2]).
(2) If $n=1$ and G is of type A_2 or B_2, then the lemma holds if any composition factor of V has a highest weight $\mu=\mu_0+p\mu_1$ such that μ_0 and $\mu_0+w\mu_1$ lie in the same alcove for any $w\in W$. Indeed, in these cases it is known that if $(Q_1(\lambda):U_1(\zeta))\neq 0$ with $\lambda\neq\zeta$, then $\langle\zeta,\alpha^\vee\rangle=p-1$ for some $\alpha\in\Delta$ (see [1, 4.2, 4.3 and 5.2]), hence ζ is p-singular. This fact implies that the composition factor $L(\zeta)$ in the proof does not occur.

Proof. (**of Theorem 1.1.**) Since $p\geq 2(h-1)$, the G_n-module $Q_n(\lambda)$ can be uniquely extended to an indecomposable G-module. By the linkage principle, the G-module $Q_n(\lambda)$ is $2(h-1)$-deep. The highest weight of $Q_n(\lambda)$ is $2(p^n-1)\rho+w_0\lambda$ and then we have

$$\langle 2(p^n-1)\rho+w_0\lambda,\alpha_0^\vee\rangle=2(p^n-1)(h-1)-\langle\lambda,\alpha_0^\vee\rangle\leq 2(p^n-1)(h-1).$$

Therefore, $Q_n(\lambda)$ satisfies all the hypotheses of V in Lemma 3.1, and we get $\mathrm{soc}^i_G Q_n(\lambda)=\mathrm{soc}^i_{G(n)}Q_n(\lambda)$ for each i. Now the result for the socle series follows from the remark of Proposition 2.2. The result for the radical series follows immediately from duality. ■

Remark. As in [9, Theorem 3.3 (2)], if $n=1$ and we assume Lusztig's conjecture for G, then $U_1(\lambda)$ is rigid for any $2(h-1)$-deep λ.

Let $M_n(\lambda)$ be a principal series module for $\lambda\in X_n$ (see [9, §1]). As in [9, Corollary 3.2], Lemma 3.1 implies the following:

Proposition 3.2. *Let $\lambda\in X_n$ and suppose that $\lambda-\rho$ is $2(h-1)$-deep. Then the Loewy length of the $G(n)$-module $M_n(\lambda)$ is equal to that of the Weyl G-module $V(\lambda+(p^n-1)\rho)$.*

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