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On the Local Structure Theorem and Equivariant Geometry of Cotangent Bundles

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Abstract. Let G be a connected reductive group acting on an irreducible normal algebraic variety X. We give a slightly improved version of Local Structure Theorems obtained by Knop and Timashev, which describe the action of some parabolic subgroup of G on an open subset of X. We also extend various results of Vinberg and Timashev on the set of horospheres in X. We construct a family of nongeneric horospheres in X and a variety $\mathcal{H}or_X$ parameterizing this family, such that there is a rational G-equivariant symplectic covering of cotangent vector bundles $T^*_{\mathcal{H}or_X} \to T^*_X$. As an application we recover the description of the image of the moment map of T^*_X obtained by Knop. In our proofs we use only geometric methods which do not involve differential operators.
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Let G be a connected reductive group acting on an irreducible normal algebraic variety X. In this paper we discuss various results describing the action of a certain parabolic subgroup of G on an open subset of X. These results are usually called "Local Structure Theorems". The first results of that kind were discovered by Grosshans [5], and independently by Brion, Luna and Vust [3]. The Local Structure Theorem was later improved by Knop [10]. He used it to integrate the invariant collective motion on the cotangent bundle T^*_X and to describe the closures in X of the so-called generic flats for the class of varieties that he called non-degenerate. (We recall the definition later.) In [18] Timashev proved a generalization of the Local Structure Theorem and this allowed him to integrate the invariant collective motion (generalizing the ideas of Knop [10]) with a weaker assumption than non-degeneracy. In this paper we give a refined version of the Local Structure Theorem obtained by Timashev. One of the applications of this theorem is to study the closures of generic flats for arbitrary varieties. This will be published elsewhere.

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The second aim of this paper is to generalize a result of Vinberg [19] who constructed a rational Galois covering of T^*_X for quasiaffine X by the cotangent bundle to the variety of generic horospheres (these results are also valid for non-degenerate varieties). By horospheres we mean the orbits of all maximal unipotent subgroups of G in X. It can be observed that the set of generic horospheres (i.e. the generic orbits of maximal unipotent subgroups of G) can be supplied with a structure of algebraic variety. The Galois group of this rational cover is equal to the little Weyl group of the variety X. It is well known that this result could not be directly generalized to arbitrary varieties since the set of generic horospheres is not good enough for this purpose, as can be seen in the case when X is a flag variety. These results were substantially generalized by Timashev in [18] for some class of varieties which is wider than non-degenerate varieties and which includes flag varieties (however this class does not contain all horospherical varieties). In fact the paper [18] contains a gap. The parabolic subgroup that Timashev denoted by Q is not well defined for all algebraic varieties. If we shrink the class of varieties for which the definition of Q from [18] is correct, the remaining arguments of Timashev are valid. In this paper we fill the gap from [18] and generalize the results mentioned above to arbitrary varieties. We construct a family of degenerate horospheres and a variety $\mathcal{H}or_X$ parameterizing them, such that there is a rational covering of the cotangent vector bundles $T^*_{\mathcal{H}or_X} \dashrightarrow T^*_X$. It will be proved that the Galois group of this rational covering is the little Weyl group introduced by Knop [9].

The structure of this paper is as follows. Section 1 is preliminary: we recall the Local Structure Theorem introduced by Knop and some consequences of it. In Section 2 we construct a Q-equivariant mapping π_D from an open subset $\mathring{X} \subset X$ to a generalized flag variety of a Levi subgroup of Q (here Q is the common stabilizer of the divisors of all B-semi-invariant rational functions, which is a parabolic subgroup of G). We want to warn the reader that our definition of Q differs from the definition given in [18]. In Section 3 we relate the fibers of the introduced map π_D to a cross section introduced by Knop. The map π_D allows us to give a refined version of the Local Structure Theorem in the sense of Timashev in Section 4. In Section 5, using ideas of Knop [12] of studying Białynicki-Birula cells, we construct a foliation of nongeneric horospheres such that the G-translate of the conormal bundle to this foliation is dense in T^*_X. We note that in the situations closely related to the topic of the present paper the Białynicki-Birula decomposition was also used by Luna [15] and Brion [2]. Section 6 is devoted to a generalization of the construction of Vinberg that relates T^*_X and the cotangent bundle to the constructed foliation of horospheres. In Section 7 we prove that the Galois group of the rational covering $T^*_{\mathcal{H}or_X} \dashrightarrow T^*_X$ is equal to the little Weyl group W_X. We also give an elementary description of the image of the normalized moment map. We note that our proof does not involve differential operators in contrast to the work of Knop [9],[12]. This work should be considered as a direct continuation of [10],[19],[18].

I am grateful to D.A.Timashev for fruitful discussions that led to simplification of most of the proofs[1] and for explaining the formula for the normal-

[1]Remark 4.4 and the ideas of the first proofs of Proposition 3.1 and Proposition 5.15, and

ized moment map. I express my gratitude to M.Brion for useful discussions and for careful reading of this work. I am grateful to F.Knop for his idea of using Białynicki-Birula cells in Section 5. I am also grateful to the referees for helpful comments that considerably improved the exposition.

Notation and conventions.

All varieties are considered over an algebraically closed field $\mathbb{K}$ of characteristic zero. By Gothic letters we denote the Lie algebras corresponding to algebraic groups denoted by the corresponding capital Latin letters. We choose a G-invariant nondegenerate symmetric bilinear form on the algebra $\mathfrak{g}$ as the trace form induced from a faithful representation of G. This form identifies $\mathfrak{g}$ and $\mathfrak{g}^*$. The action $G : \mathfrak{g}$ (resp. $G : \mathfrak{g}^*$) is always assumed to be (co)adjoint. For a subspace $\mathfrak{h} \subset \mathfrak{g}$, by $\mathfrak{h}^\perp$ we denote the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$.

We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let B^- be the unique Borel subgroup of G such that $B^- \cap B = T$. By $P \supset B$ we denote a parabolic subgroup. By $P^- \supset B^-$ we denote the parabolic subgroup opposite to P. Denote by P_u (resp. P_u^-) the unipotent radical of P (resp. P^-).

Let $\Xi = \Xi(T)$ be the character lattice of T and let $\Lambda = \Lambda(T)$ be the lattice of one-parameter subgroups of T. For a one-parameter subgroup $\lambda\colon \mathbb{K}^* \to T$ and a character $\chi \in \Xi$ we have a pairing $\langle \lambda, \chi \rangle$ defined by the formula $\chi(\lambda(t)) = t^{\langle \lambda, \chi \rangle}$ which identifies Λ and Ξ^*. We use the additive notation for the group law in Λ and Ξ. For $\chi \in \Xi$ its differential $d\chi$ defines the linear function on $\mathfrak{t}$ which we denote by the same letter.

We denote by $W = N_G(T)/T$ the Weyl group of G. Let Δ be the root system of the Lie algebra $\mathfrak{g}$ corresponding to T, let $\Delta^+(\Delta^-)$ be the system of positive (negative) roots corresponding to the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, and let Π be the system of simple roots of Δ^+. We also have the standard decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ into the root subspaces. For $\alpha \in \Delta$ let $e_\alpha \in \mathfrak{g}_\alpha$ be the corresponding element of a Chevalley basis, $\alpha^\vee$ be the corresponding coroot, and s_α be the corresponding reflection. $w_0 \in W$ is the longest element in the Weyl group. By $\mathfrak{t}^*/W$ we denote the geometric quotient of $\mathfrak{t}^*$ by W. For a parabolic subgroup $P \supset B$ we denote by L its Levi subgroup containing T. Then $B_L = L \cap B$ is a Borel subgroup of L. Let $\Delta_L \subset \Delta$ and $\Delta_L^+ \subset \Delta_L$ be the root subsystem corresponding to L and its set of positive roots, respectively. The subset of simple roots in Δ_L^+ is denoted by Π_L. We denote by $C_L \subset \Xi \otimes_{\mathbb{Z}} \mathbb{Q}$ the dominant Weyl chamber of L with respect to the positive root system Δ_L^+. By C_L° we denote the interior of C_L. For a parabolic subgroup P (resp. for a parabolic subalgebra $\mathfrak{p}$) containing T (resp. $\mathfrak{t}$) we denote by Δ_{P_u} (resp. $\Delta_{\mathfrak{p}_u}$) the subset of roots in Δ corresponding to the root decomposition of $\mathfrak{p}_u$.

Let V_χ be the simple G-module with highest weight χ and let V_χ^* be its dual . A highest weight vector of V_χ is denoted by σ_χ and a lowest weight vector of V_χ^* is denoted by $\sigma_{-\chi}^*$, $\langle v, w \rangle$ is the pairing of $v \in V_\chi$ and $w \in V_\chi^*$. By $\mathrm{Wt}(V_\chi)$ we denote the set of weights for the T-action on V_χ.

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the idea how to prove Theorem 5.20 in a stronger form are due to D.A.Timashev, who kindly proposed them after reading a preliminary version of this paper.

For an algebraic group H, by the superscript $(-)^{(H)}$ we denote H-semi-invariants and by $(-)^{(H)}_{\chi}$ we denote H-semi-invariants of weight χ.

Let $G \supset H$ be linear algebraic groups and Z be a quasiprojective H-variety. We may form a quasi-projective G-variety $G *_H Z$, being the quotient of $G \times Z$ by the action of H: $(g,z) \mapsto (gh^{-1}, hz)$. The image of a point (g,z) in this quotient is denoted by $g * z$.

For an algebraic group action G on X, ξx is the velocity vector of $\xi \in \mathfrak{g}$ at $x \in X$, $\mathfrak{g}x$ is the tangent space to the orbit Gx in x and G_x is the stabilizer of x. For affine X, in the case when the algebra $\mathbb{K}[X]^G$ of G-invariant regular functions on X is finitely generated, by $X/\!\!/G$ we denote the categorical quotient of X, which is isomorphic to Spec $\mathbb{K}[X]^G$. If the variety X is smooth, we can define the moment map $\mu_X : T^*_X \longrightarrow \mathfrak{g}^*$ (where T^*_X is the cotangent bundle of X) by the following formula:

$$\langle \mu_X(\alpha), \xi \rangle = \langle \alpha, \xi x \rangle, \ \forall x \in X, \ \alpha \in T_{X,x}, \ \xi \in \mathfrak{g}.$$

We recall that for a homogeneous variety $X = G/H$ the cotangent bundle T^*_X can be expressed as

$$T^*_X \cong G *_H (\mathfrak{g}/\mathfrak{h})^* \cong G *_H \mathfrak{h}^{\perp}.$$

In this case the moment map is induced by the inclusion $\mathfrak{h}^{\perp} \hookrightarrow \mathfrak{g}^*$ and its image is equal to $G\mathfrak{h}^{\perp}$.

1. Local Structure Theorem

We begin with some preliminary remarks. Consider a normal G-variety X and a B-stable Weil divisor $D = \sum a_i D_i$, where D_i are B-stable prime Weil divisors. Let us call such D a B-divisor for brevity. We denote by $P[D_i]$ the stabilizer in G of D_i. The stabilizer of the B-divisor D is defined as the intersection of the stabilizers of its prime components $P[D] = \bigcap_{a_i \neq 0} P[D_i]$, which is a parabolic subgroup of G. Since the number of the parabolic subgroups containing B is finite, there exists a B-divisor for which $P[D]$ is absolutely minimal. We denote this parabolic subgroup by $P(X)$. Let us recall the following lemma.

Lemma 1.1. *([10, Lemma 2.2]) Let X be a normal G-variety and $D \subset X$ a prime divisor. Then D is a Cartier divisor outside $Y = \bigcap_{g \in G} gD$.*

Later for D we shall take a B-invariant but not G-invariant divisor. Thus Y is a proper subset of D, so considering $X \setminus Y$ instead of X, we can assume that D is Cartier.

Replacing a Cartier divisor D with a sufficiently large multiple nD, we may assume that D is G-linearized ([8]), and in particular B-linearized. Any two G-linearizations differ by a character of G, we choose one of them. For the divisor nD consider the canonical B-semi-invariant rational section σ_{nD} of the associated line bundle $\mathcal{O}(nD)$ of weight χ_{nD}. Let us put $\chi_D = \chi_{nD}/n$.

Definition 1.2. ([10]) Consider a B-divisor D of weight χ_D. The weight χ_D is called $P[D]$-regular if $\langle \chi_D, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Delta_{P[D]_u}$.

Remark 1.3. We recall that $\langle \chi_D, \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta_{L[D]}$, where $L[D] \subset P[D]$ is the Levi subgroup containing T. We also note that the weight of every effective divisor D with stabilizer $P[D]$ is $P[D]$-regular. This is a standard fact of representation theory applied to the B-semi-invariant section $\sigma_D \in H^0(X, \mathcal{O}(D))^{(B)}_{\chi_D}$ which is a highest weight vector.

Let us recall from [10] the definition of non-degenerate varieties:

Definition 1.4. A G-variety X is called non-degenerate if there exists a rational B-semi-invariant function $f_\chi \in \mathbb{K}(X)^{(B)}_\chi$ with divisor $D = (f_\chi)$ such that χ is $P(X)$-regular.

Remark 1.5. We note that a quasiaffine variety is non-degenerate.

Given a B-divisor, we consider the following $P[D]$-equivariant map ([10]):

$$\psi_D : X \setminus D \longrightarrow \mathfrak{g}^*, \ x \mapsto l_x, \ \ \textit{where} \ \ l_x(\xi) = \frac{\xi \sigma_D}{\sigma_D}(x).$$

Let us recall the version of the Local Structure Theorem obtained by Knop.

Theorem 1.6. *([10, Thm. 2.3, Prop. 2.4]) Let X be a normal G-variety with a B-divisor D. Assume that χ_D is $P[D]$-regular. For some $x_0 \in X \setminus D$ let $\eta_0 := \psi_D(x_0)$, $L := G_{\eta_0}$ and $Z := \psi_D^{-1}(\eta_0)$. Then:*

(i) The image of ψ_D is a single $P[D]$-orbit equal to $\eta_0 + \mathfrak{p}[D]_u$, and the restriction of the linear function η_0 to $\mathfrak{p}[D]$ is equal to χ_D.

(ii) L is a Levi subgroup of $P[D]$ and there is an isomorphism

$$P[D] *_L Z \longrightarrow X \setminus D$$

(iii) Suppose that $P[D] = P(X)$. Then the kernel L_0 of the action of L on Z contains the commutator subgroup $[L, L]$.

For simplicity we denote $P(X)$ by P. Let us notice that in the theorem x_0 can be chosen so that $L \supset T$.

In the situation of Theorem 1.6 (iii), we see that the torus $A := L/L_0 = P/L_0P_u$ is acting effectively on Z (The group L_0P_u is denoted by P_0). And from $\mathbb{K}(X)^{(B)} = \mathbb{K}(Z)^{(B_L)} = \mathbb{K}(Z)^{(L)}$ one can identify $\Xi(A)$ with the group of characters

$$\Xi(X) = \{\chi \mid \mathbb{K}(X)^{(B)}_\chi \neq 0\}.$$

We shall write $A = A_X$ (resp. $\mathfrak{a} = \mathfrak{a}_X$) if we want to stress the dependence on the variety X. Let us recall that as a consequence of the Local Structure Theorem we get that general P_0-orbits coincide with general P_u-orbits.

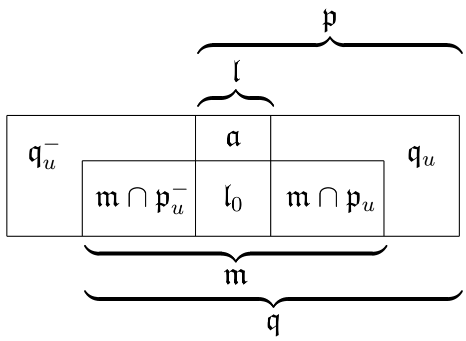
2. Equivariant maps to flag varieties

To formulate a refined version of the Local Structure Theorem we introduce some additional notation.

We denote by Q some parabolic subgroup of G containing P. Let M be the Levi subgroup of Q containing the maximal torus T. We have a Levi decomposition $Q = Q_u \rtimes M$. Let us assume that $T \cap [M, M] \subset L_0$. Later in the proof of Theorem 4.1 we shall choose Q to be the common stabilizer of the divisors of all B-semi-invariant functions; then it satisfies this property.

It is easy to see that $Q_u \subset P_u$ and $L \subset M$. Consider the group $M_0 = [M, M]Z(L_0)$. We also put $Q_0 = Q_u \rtimes M_0$, so we have $A \cong M/M_0 \cong Q/Q_0$.

Let us embed $\mathfrak{a}$ into $\mathfrak{l}$ as the orthocomplement to $\mathfrak{l}_0$. Then the group $Z_G(\mathfrak{a})$ contains M. We can describe the relations among the introduced Lie algebras by the following picture taken from [18].



Identifying $\mathfrak{g} \cong \mathfrak{g}^*$ via the invariant bilinear form fixed in the conventions we see that $\mathfrak{l}, \mathfrak{l}_0, \mathfrak{m}, \mathfrak{m}_0, \mathfrak{a}$ are self-dual. Also we have $\mathfrak{p}_u = \mathfrak{p}^\perp \cong (\mathfrak{g}/\mathfrak{p})^* \cong (\mathfrak{p}_u^-)^*$, and $\mathfrak{q}_u = \mathfrak{q}^\perp \cong (\mathfrak{g}/\mathfrak{q})^* \cong (\mathfrak{q}_u^-)^*$. Let us denote

$$\mathfrak{a}^{pr} := \{\xi \in \mathfrak{a} |\ Z_G(\xi) = Z_G(\mathfrak{a}),\ g\xi \notin \mathfrak{a} \text{ for all } g \in G \setminus N_G(\mathfrak{a})\},$$

in fact $\mathfrak{a}^{pr}$ is obtained from $\mathfrak{a}$ by throwing away a finite union of subspaces.

We shall construct the morphism that is the main tool in the proof of the refined Local Structure Theorem in the sense of Timashev.

Let us fix an effective G-linearized B-divisor D and the corresponding section $\sigma_\chi = \sigma_D$ which is a highest weight vector of weight $\chi = \chi_D$. Consider the action of M on the space of sections $H^0(X, \mathcal{O}(D))$. Let $V_\chi(M) := \langle M\sigma_\chi \rangle$ be the M-module generated by σ_χ. It is simple since σ_χ is B-semi-invariant. $V_\chi(M)$ can be considered as a simple Q-module fixed pointwise by Q_u. Indeed Q_u is a normal subgroup in Q which stabilizes the highest weight vector σ_χ. Moreover $Z(M)$ acts by a character on the simple module $V_\chi(M)$.

Let $|V_\chi(M)|$ be the linear system on X (possibly not complete) corresponding to the Q-module $V_\chi(M) \subset H^0(X, \mathcal{O}(D))$.

Remark 2.1. The basepoint set of the linear system $|V_\chi(M)|$ is M-invariant and is equal to $\bigcap_{m \in M} mD$ since $V_\chi(M)$ is the linear span of the M-orbit of σ_D.

Consider the morphism:

$$\pi_D\colon X\setminus\bigcap_{m\in M} mD\longrightarrow \mathbb{P}(V_\chi(M)^*),$$

defined by the linear system $|V_\chi(M)|$. It is easy to see that π_D is Q-equivariant. This implies the following lemma.

Lemma 2.2. *Any orbit of the radical $Z(M)\ltimes Q_u$ of Q is contained in a fiber of π_D.*

Remark 2.3. If the divisor D is M-invariant, then $\mathbb{P}(V_\chi(M)^*)$ is a point. For our purposes it is sufficient to consider a divisor that is not M-stable. This condition implies that $\operatorname{codim}\bigcap_{m\in M} mD\geqslant 2$.

Now we are ready to state one of the main theorems of the paper.

Theorem 2.4. *Let D be an effective G-linearized B-divisor with the canonical section $\sigma_D\in H^0(X,\mathcal{O}(D))^{(B)}_\chi$ and let $P[D]$ be the stabilizer of D. Consider a parabolic subgroup Q of G containing P with Levi subgroup M containing T that satisfies the inclusion $T\cap[M,M]\subset L_0$. Then the image of the morphism*

$$\pi_D\colon \mathring{X}=X\setminus(\bigcap_{m\in M} mD)\longrightarrow\mathbb{P}(V_\chi(M)^*)$$

coincides with the flag variety $M\langle\sigma^*_{-\chi}\rangle\cong M/P^-_M$, where $P^-_M=M\cap P[D]^-$.

Proof. First let us recall that $M\langle\sigma^*_{-\chi}\rangle$ is the unique closed orbit in $\mathbb{P}(V_\chi(M)^*)$. We begin with the following well known lemmas.

Lemma 2.5. *Let $\lambda\in C^\circ_M$. Denote by $\operatorname{Ann}\sigma_\chi$ the hyperplane in $V_\chi(M)^*$ annihilating σ_χ. Then for any point $x\in\mathbb{P}(V_\chi(M)^*)\setminus\mathbb{P}(\operatorname{Ann}\sigma_\chi)$ we have $\lim_{t\to 0}\lambda(t)x=\langle\sigma^*_{-\chi}\rangle$, besides $Mx\not\subseteq\mathbb{P}(\operatorname{Ann}\sigma_\chi)$ for any $x\in\mathbb{P}(V_\chi(M)^*)$.*

Proof. Let $v\in V_\chi(M)^*$ be a vector representing x. Then $\langle Mv\rangle=V_\chi(M)^*$ by the simplicity of $V_\chi(M)^*$, in particular $Mx\not\subseteq\mathbb{P}(\operatorname{Ann}\sigma_\chi)$. For $v\in V_\chi(M)^*\setminus\operatorname{Ann}\sigma_\chi$ consider the decomposition in the sum of weight vectors:

$$v=c\sigma^*_{-\chi}+\sum_{\substack{\omega\in\mathrm{Wt}(V_\chi(M)^*)\\ \omega\neq-\chi}} c_\omega\sigma^*_\omega,$$

where $\sigma^*_\omega\in V_\chi(M)^*$ is a vector of weight ω and $c\neq 0$. Since $-\langle\lambda;\chi\rangle<\langle\lambda;\omega\rangle$, for $\lambda\in C^0_M$ and every $\omega\in\mathrm{Wt}(V_\chi(M)^*)$ which is distinct from $-\chi$, we obtain:

$$\lim_{t\to0}\lambda(t)x=\lim_{t\to0}\left\langle\sigma^*_{-\chi}+\sum_{\substack{\omega\in\mathrm{Wt}(V_\chi(M)^*)\\ \omega\neq-\chi}} t^{\langle\lambda,\chi+\omega\rangle}\frac{c_\omega}{c}\sigma^*_\omega\right\rangle=\langle\sigma^*_{-\chi}\rangle. \qquad\blacksquare$$

Let us recall the following lemma.

Lemma 2.6. *[17, Prop. 8.4.5] Let G be a reductive group, T be a maximal torus and P be some parabolic subgroup containing T. Consider a one-parameter subgroup $\lambda \in \Lambda(T)$ such that $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Delta_{P_u}$. Then we have $\lim_{t\to 0} \lambda(t) p_u \lambda(t)^{-1} = e$ for $p_u \in P_u$.*

We note that by definition of π_D the image $\pi_D(X \setminus D)$ is equal to the complement of $\mathbb{P}(\operatorname{Ann} \sigma_\chi)$ in $\pi_D(\mathring{X})$. We also have

$$X \setminus D = \pi_D^{-1}(\mathbb{P}(V_\chi(M)^*) \setminus \mathbb{P}(\operatorname{Ann} \sigma_\chi)).$$

According to the Local Structure Theorem 1.6 we may choose a dense open B-invariant subset X° of $X \setminus D$ isomorphic to $P *_L Z^\circ$, where L_0 is acting trivially on Z°. Let us choose a one-parameter subgroup λ of the torus $[M, M] \cap T$ such that $\lambda \in C_M^\circ$. Since $[M, M] \cap T \subset L_0$, the one-parameter subgroup λ acts trivially on Z°. Let us calculate the limit $\lim_{t\to 0} \lambda(t) x$ for $x \in \pi_D(X^\circ)$ in two different ways. Since $\pi_D(X^\circ) \subset \mathbb{P}(V_\chi(M)^*) \setminus \mathbb{P}(\operatorname{Ann} \sigma_\chi)$ Lemma 2.5 implies the following.

Lemma 2.7. *For $x \in \pi_D(X^\circ)$ we have $\lim_{t\to 0} \lambda(t) x = \langle \sigma^*_{-\chi} \rangle$.*

Let us calculate this limit in a different way. By the Local Structure Theorem the action of P_u is free on X° and we have $X^\circ = P_u Z^\circ$.

Lemma 2.8. *Let λ be a one-parameter subgroup of the torus $[M, M] \cap T$ such that $\lambda \in C_M^\circ$. Then for $x = \pi_D(p_u z) \in \pi_D(X^\circ)$, where $p_u \in P_u$, $z \in Z^\circ$, we have $\lim_{t\to 0} \lambda(t) x = \pi_D(z)$.*

Proof. Consider the decomposition $Q = Q_u \rtimes M$; combining it with the inclusions $Q_u \subset P_u \subset Q$ we get $P_u = Q_u \rtimes (M \cap P_u)$. Thus we have $p_u = q_u m$, for $q_u \in Q_u$, $m \in M \cap P_u$. We obtain

$$x = \pi_D(p_u z) = \pi_D(q_u m z) = m \pi_D(z).$$

Since $\lambda \in C_M^\circ$ is positive on all roots corresponding to $M \cap P_u$, by Lemma 2.6 we get $\lim_{t\to 0} \lambda(t) m \lambda(t)^{-1} = e$ for $m \in M \cap P_u$. The triviality of the action of λ on Z° implies that

$$\lim_{t\to 0} \lambda(t) x = (\lim_{t\to 0} \lambda(t) m \lambda(t)^{-1}) \pi_D(z) = \pi_D(z). \qquad \blacksquare$$

Combining Lemma 2.7 and Lemma 2.8 we get $\pi_D(Z^\circ) = \langle \sigma^*_{-\chi} \rangle$. Using the fact that $\pi_D(X^\circ) = (M \cap P_u) \pi_D(Z^\circ)$, we obtain that

$$M \pi_D(X^\circ) = M \pi_D(Z^\circ) = M \langle \sigma^*_{-\chi} \rangle.$$

Since $M \pi_D(X^\circ)$ is dense in $\pi_D(\mathring{X})$, this proves our theorem. $\blacksquare$

Corollary 2.9. *(of the proof of Theorem 1.9) Let X° be the dense open B-invariant subset of $X \setminus D$ isomorphic to $P *_L Z^\circ$, where L_0 is acting trivially on Z°. The section Z° is contained in a fiber of the map π_D.*

Now we are able to describe the sets $X \setminus D$ and D as preimages for the map π_D of some subsets in M/P_M^-.

Proposition 2.10. *The set $X \setminus D$ is the preimage under π_D of the open Bruhat cell $B_M P_M^-/P_M^- \subset M/P_M^-$, where $B_M = B \cap M$. The set $D \cap \mathring{X}$ is equal to the preimage of the complement of this open cell in M/P_M^-.*

Proof. Let us recall that the complement of the open cell in $M\langle\sigma_{-\chi}^*\rangle \subset \mathbb{P}(V_\chi(M)^*)$ can be described as

$$\{\langle\sigma^*\rangle \in M/P_M^- \mid \langle\sigma^*, \sigma_\chi\rangle = 0\}. \qquad (*)$$

We see that equality $\sigma_\chi(x) = 0$ (i.e. $x \in D$) is equivalent to $\langle\pi_D(x), \sigma_\chi\rangle = 0$. Thus we get that $X \setminus D$ maps to the open cell $B_M P_M^-/P_M^- \subset M/P_M^-$ and $D \cap \mathring{X}$ maps to the complement of the open cell. Since $\pi_D : \mathring{X} \to M/P_M^-$ is surjective this proves our proposition. ∎

3. Relation between π_D and cross sections.

Now we state a result which relates the cross sections from the Local Structure Theorem, introduced by Knop, and the fibers of π_D.

Proposition 3.1. *Consider an effective G-linearized B-divisor D with weight χ and the canonical section σ_χ. We have a map $\psi_D : X \setminus D \longrightarrow \mathfrak{g}^*$, $x \mapsto l_x$, where $l_x(\xi) = \frac{\xi\sigma_\chi}{\sigma_\chi}(x)$. For a point $x_0 \in X \setminus D$ consider $Z := \psi_D^{-1}(\psi_D(x_0))$. Let π_D be the map constructed in Theorem 2.4 for the divisor D. Then*

$$Z \subset \pi_D^{-1}(\pi_D(x_0)).$$

We shall give two proofs of this proposition based on different observations.

Proof. [First proof of Proposition 3.1] Denote by $L[D]$ the Levi subgroup of $P[D]$ that is the stabilizer of $\psi_D(x_0)$. By Q-equivariance of π_D we have $p_u Z \subset \pi_D^{-1}(\pi_D(p_u x_0))$ for $p_u \in P[D]_u$. After translating x_0 by p_u we can assume that $L[D] \supset T$. Since $P[D] \supset P$ this also implies $L[D] \supset L$. By the Local Structure Theorem we have $X \setminus D = P[D] *_{L[D]} Z$. We can apply the Local Structure Theorem to the action $L[D] : Z$ and to the divisor $D_0 \cap Z$ where $D_0 \subset X$ is a B-divisor with stabilizer $P := P(X)$. Thus we get that Z contains an open subset isomorphic to $(L[D] \cap P) *_L Z_0$ where the action $L_0 : Z_0$ is trivial. By Corollary 2.9 we know that $\pi_D(Z_0) = e(M \cap P[D]^-)$. Let us prove that the group $L[D] \cap P_u$ fixes this point. Indeed, from the decomposition $P_u = (P_u \cap M)Q_u$ using the root decompositions with respect to T of the corresponding Lie algebras we get $L[D] \cap P_u = (L[D] \cap M \cap P_u)(L[D] \cap Q_u)$. The claim follows from the facts that Q_u-orbits lie in the fibers of π_D and that $L[D] \cap M$ fixes $e(M \cap P[D]^-)$. This implies that $\pi_D((L[D] \cap P_u)Z_0) = e(M \cap P[D]^-)$, which proves the proposition. ∎

Proof. [Second proof of Proposition 3.1] The set $\pi_D(X \setminus D)$ is equal to the open $P[D]_u \cap M$-orbit in $M/(M \cap P[D]^-)$. If the statement of the proposition were not true, there would exist a point $x \in Z$ such that $\pi_D(x) \neq \pi_D(x_0)$ and $\pi_D(x) = p\pi_D(x_0)$ for some $p \in P[D]_u \cap M$.

Consider the restriction map

$$\psi_D|_{\mathfrak{m}} : X \setminus D \longrightarrow \mathfrak{g}^* \longrightarrow \mathfrak{m}^*,$$

$$\psi_D|_{\mathfrak{m}} : x \mapsto l_x, \quad where \quad l_x(\xi) = \frac{\xi\sigma_\chi}{\sigma_\chi}(x), \ \forall \xi \in \mathfrak{m}.$$

The map ψ_D is $P[D]$-equivariant, and this implies $P[D] \cap M$-equivariance of $\psi_D|_{\mathfrak{m}}$. We have the following evident lemma.

Lemma 3.2. *The map $\psi_D|_{\mathfrak{m}} \colon X \setminus D \to \mathfrak{m}^*$ is equal to the composition of the map $\pi_D \colon \mathring{X} \to \mathbb{P}(V_\chi(M)^*)$ and the map $\mathbb{P}(V_\chi(M)^*) \setminus \mathbb{P}(\operatorname{Ann} \sigma_\chi) \to \mathfrak{m}^*$ defined as*

$$\langle \sigma^* \rangle \mapsto l_{\langle \sigma^* \rangle}, \quad where \quad l_{\sigma^*}(\xi) = \frac{\langle \xi\sigma_\chi, \sigma^* \rangle}{\langle \sigma_\chi, \sigma^* \rangle}, \ \forall \xi \in \mathfrak{m}.$$

Applying Theorem 1.6 we obtain that the image of $\psi_D|_{\mathfrak{m}}(X \setminus D)$ is a single $P[D]_u \cap M$-orbit and the stabilizer of $\psi_D|_{\mathfrak{m}}(x_0)$ in $P[D]_u \cap M$ is trivial. By Lemma 3.2 the equality $\pi_D(x) = p\pi_D(x_0) = \pi_D(px_0)$ for $p \in P_u \cap M$ implies that $\psi_D|_{\mathfrak{m}}(x) = \psi_D|_{\mathfrak{m}}(px_0)$. Taking into account that

$$\psi_D|_{\mathfrak{m}}(x_0) = \psi_D|_{\mathfrak{m}}(x) = \psi_D|_{\mathfrak{m}}(px_0) = p\psi_D|_{\mathfrak{m}}(x_0),$$

we come to a contradiction since p does not stabilize $\psi_D|_{\mathfrak{m}}(x_0)$. ■

Proposition 3.3. *The fibers of the map π_D are irreducible. The set $\pi_D^{-1}(\pi_D(x_0))$ is identified with*

$$(L[D] \cap Q) \ltimes (P[D]_u \cap Q_u) *_{L[D] \cap Q} Z \cong (P[D]_u \cap Q_u) \times Z.$$

Proof. Let $x \in M/P_M^-$ be the point corresponding to the right coset eP_M^-. The preimage $\pi_D^{-1}((P[D]_u \cap M)x)$ of the open cell in M/P_M^- is irreducible, being the dense open subset $X \setminus D$ in X. It is isomorphic to $P[D]_u \cap M \times \pi_D^{-1}(x)$, which implies the irreducibility of $\pi_D^{-1}(x)$. Since the fibers of π_D are permuted transitively via the action of M this proves the first part of the proposition. The second part follows immediately from the isomorphism $X \setminus D \cong P[D]_u \cap M \times P[D]_u \cap Q_u \times Z$, the inclusion $Z \subset \pi_D^{-1}(\pi_D(x_0))$ and the freeness of the action of $P[D]_u \cap M$ on the open cell in M/P_M^-. ■

Consider the set of fundamental weights $\{\omega_\alpha\}$ of $[M, M]$ where $\alpha \in \Pi_M$.

Proposition 3.4. *Let α be a simple root in $\Pi_M \setminus \Pi_L$ and ω_α be the corresponding fundamental weight of M. Then there exists a unique prime B-divisor $D_\alpha \subset X$ such that the restriction of the weight χ_{D_α} to $[M, M] \cap T$ is equal to ω_α. Let D be any B-divisor. Then $\chi_D|_{[M,M] \cap T} = \sum_{\alpha \in \Pi_M \setminus \Pi_L} \langle \chi_D, \alpha^\vee \rangle \omega_\alpha$ and $D = \sum_{\alpha \in \Pi_M \setminus \Pi_L} \langle \chi_D, \alpha^\vee \rangle D_\alpha + D_Q$ for some Q-invariant divisor D_Q.*

Proof. Consider an effective B-invariant divisor D_0 such that $P[D_0] = P$ and let us denote $P_M = P \cap M$. For a divisor $F_\alpha = \overline{B_M s_\alpha P_M^-/P_M^-}$ (with the stabilizer P_α in M, which is a maximal parabolic subgroup of M) let us choose a natural number n such that nF_α is M-linearized. By Proposition 3.3 the B-divisor $D_\alpha := \pi_{D_0}^{-1}(F_\alpha)$ is irreducible. Let us notice that nD_α is the zero divisor of the section of the M-linearized line bundle $\pi_{D_0}^*(\mathcal{O}(nF_\alpha))$, which is the pullback of the $B \cap M$-semi-invariant section σ_{nF_α}. We also have $\chi_{D_\alpha}|_{[M,M]\cap T} = \omega_\alpha$.

Let D' be a prime B-invariant divisor that is not M-invariant. Since $\pi_{D'}(D')$ is prime, by Proposition 2.10 it is equal to a prime Schubert divisor in $M/M \cap P[D']^-$ of weight ω_α for some $\alpha \in \Pi_M \setminus \Pi_L$. Thus $\chi_{D'}|_{[M,M]\cap T} = \omega_\alpha$.

To prove the uniqueness of D_α assume that we have two distinct prime divisors D_α and D'_α such that the restrictions to $[M,M] \cap T$ of χ_{D_α} and $\chi_{D'_\alpha}$ are equal to ω_α. This implies that the stabilizer in M of $D_\alpha + D'_\alpha$ is equal to P_α. Consider the map $\pi_\alpha := \pi_{D_\alpha + D'_\alpha}$. By Theorem 2.4 the image of π_α is equal to M/P_α^- and, by Proposition 2.10, $D_\alpha \cup D'_\alpha$ is the preimage of the unique B_M-invariant divisor in M/P_α^-. This contradicts the irreducibility of the fibers of π_α and proves the first part of the corollary.

From the above we get a decomposition $D = \sum_{\alpha\in\Pi_M\setminus\Pi_L} n_\alpha D_\alpha + D_Q$ for some Q-invariant D_Q and integer n_α. Comparing the $([M,M]\cap T)$-weights of both sides of this equality and using the equality $\chi_{D_\alpha}|_{[M,M]\cap T} = \omega_\alpha$, we get $\chi_D|_{[M,M]\cap T} = \sum_{\alpha\in\Pi_M\setminus\Pi_L} n_\alpha\omega_\alpha$. Since the set of coroots $\alpha^\vee$ for $\alpha \in \Pi_M$ form the dual basis to the basis formed by the fundamental weights of $[M,M]$, we have $n_\alpha = \langle \chi_D, \alpha^\vee \rangle$. ∎

Corollary 3.5. *Let D be a B-divisor such that its weight χ_D has zero restriction to $[M,M] \cap T$. Then the divisor D is Q-invariant. In particular, every B-semi-invariant rational function on X is Q-semi-invariant.*

4. Refined Local Structure Theorem

Using Theorem 2.4 we shall derive a stronger version of the Local Structure Theorem in the sense of Timashev [18, Thm. 3]. Let us first introduce some notation.

We are going to choose a parabolic subgroup $Q \supset P$ with Levi subroup M, for which $[M,M] \cap T \subset L_0$. Consider the set $\mathcal{E}$ of prime B-divisors that occur in the divisors (f) of rational B-semi-invariant functions $f \in k(X)^{(B)}$. In other words,

$$\mathcal{E} := \{E \mid \exists f \in k(X)^{(B)},\ (f) = \sum a_D D,\ a_E \neq 0\}.$$

We denote by $Q \supset P$ the parabolic subgroup that is the common stabilizer of all B-divisors from $\mathcal{E}$. Since Q stabilizes the divisor (f) for $f \in k(X)^{(B)}$, by a consequence of a theorem of Rosenlicht [20, Thm. 3.1] we obtain that f is Q-semi-invariant. Let M be the Levi subgroup of Q containing the maximal torus T. We note that $T \cap [M,M] \subset L_0$, since $T \cap [M,M]$ is acting trivially on $f \in k(X)^{(B)}$. By Corollary 3.5 this parabolic subgroup Q is maximal in the set of parabolic subgroups stabilizing some B-divisor (in particular containing P) and satisfying the property $T \cap [M,M] \subset L_0$.

We choose an effective B-divisor D such that the stabilizer of D in Q is equal to P. Let us also choose an effective divisor $E \in \mathcal{E}$ such that Q is the stabilizer of E in G. In particular, the stabilizer of $D \cup E$ is equal to P. Let $\sigma_E \in H^0(X, \mathcal{O}(E))^{(Q)}$ be the Q-semi-invariant section that defines E. Since E is effective the weight χ_E of the divisor E is Q-regular. By X_1 let us denote the open Q-invariant subset $\mathring{X} \setminus E$ and by X° the P-invariant subset $X \setminus (D \cup E)$. We recall that $\mathring{X} = X \setminus \bigcap_{m \in M} mD$. We also consider the map:

$$\psi_E : \ X \setminus E \longrightarrow \mathfrak{g}^*, \ x \mapsto l_x(\xi) = \frac{\xi \sigma_E}{\sigma_E}(x).$$

Theorem 4.1. *There exists a point $x_0 \in X^\circ$ such that $M_{\pi_D(x_0)} = P_M^-$ and M is the stabilizer of $\psi_E(x_0)$. For the M-stable closed subset $Z_1 := \psi_E^{-1}\psi_E(x_0)$ of X_1 we have a Q-equivariant isomorphism*

$$Q *_M Z_1 \longrightarrow X_1.$$

Moreover, $Z_1 \cong M/P_M^- \times Z_0$ for $Z_0 := \pi_D^{-1}\pi_D(x_0) \cap Z_1$. Here M acts on the product $M/P_M^- \times Z_0$ by left multiplication on M/P_M^- and via the quotient $M/M_0 \cong A$ on Z_0.

Proof. Consider a point $x_0 \in X^\circ$; since the Levi subgroups of Q are conjugate by the elements of Q_u, translating x_0 by an element of Q_u we may assume that M is the stabilizer of $\psi_E(x_0)$. By Proposition 2.10, the variety X° is the preimage under π_D of the open Bruhat cell $(P_u \cap M)P_M^-/P_M^-$ in M/P_M^-. In particular translating x_0 by an element of $(P_u \cap M)$ we can assume that x_0 is in X°, the stabilizer of $\psi_E(x_0)$ is still M and $M_{\pi_D(x_0)} = P_M^-$.

The isomorphisms $Q *_M Z_1 \longrightarrow X_1$ and $Q *_M Z_1 \cong Q_u \times Z_1$ follow from the Local Structure Theorem 1.6 applied to the point $x_0 \in X^\circ$, the divisor E and its stabilizer Q. From the M-equivariance of π_D and M-invariance of Z_1 we see that $Z_1 = MZ_0$.

Lemma 4.2. *We have the isomorphism $P *_L Z_0 \xrightarrow{\sim} X^\circ$, where the group L acts on Z_0 via the quotient L/L_0.*

Proof. Let us notice that $\pi_D(X^\circ) \supset \pi_D(Z_1 \cap X^\circ) \supset P_M \pi_D(x_0) \cong P_M/L$. This implies that the variety $Z_1 \cap X^\circ$ (which is equal to $Z_1 \setminus D$) projects P_M-equivariantly onto the open cell $\pi_D(X^\circ) = (P_u \cap M)P_M^-/P_M^-$, which is isomorphic to P_M/L, and in particular $Z_1 \setminus D \cong P_M *_L Z_0$.

The action of Q_u on X° is free, which implies $X^\circ \cong Q_u \times (Z_1 \setminus D)$. Since $P = Q_u \rtimes P_M$ we have

$$X^\circ \cong P *_{P_M} (Z_1 \setminus D) \cong P *_L Z_0.$$

Let us apply the Local Structure Theorem to the effective divisor $D \cup E$ and the group P. We get that $X^\circ \cong P *_L Z_0'$ for some Z_0' with trivial action of L_0. The L-equivariant isomorphisms $X^\circ/P_u \cong Z_0' \cong Z_0$ imply the triviality of the action of L_0 on Z_0, which proves the lemma. ■

We shall prove the isomorphism $Z_1 \cong M/P_M^- \times Z_0$ in several steps. Let us first prove that the $P_u \cap M$-orbit of each point $z \in Z_1 \cap X^\circ$ is contained as a dense open subset in the orbit $M_0 z$. This was first noticed by Timashev [18] in less general settings for a general point of Z_1.

Step 1. Since the divisor of any B-semi-invariant rational function $f \in \mathbb{K}(X)^{(B)}$ is Q-invariant, as noted before this function should be Q-semi-invariant as well. Thus we get $\mathbb{K}(X)^{(B)} = \mathbb{K}(X)^{(P)} = \mathbb{K}(X)^{(Q)}$. Since $\mathbb{K}(X)^U$ is generated by $\mathbb{K}(X)^{(B)}$, these equalities imply that $\mathbb{K}(X)^U = \mathbb{K}(X)^{P_0} = \mathbb{K}(X)^{Q_0}$. We shall prove that a general Q_0-orbit contains a general P_0-orbit as a dense open subset. Assume the converse. Then a general Q_0-orbit contains an infinite family of P_0-orbits. Due to the Rosenlicht Theorem general P_0-orbits are separated by rational invariants from $k(X)^{P_0}$. This implies that there exists an invariant $f \in k(X)^{P_0}$ which has non-constant value on the family of P_0-orbits contained in a general Q_0-orbit, which contradicts the inclusion $f \in k(X)^{Q_0}$.

Step 2. The isomorphism $Q *_M Z_1 \cong Q_u \times Z_1 \longrightarrow X_1$ implies that for $z \in Z_1$ we have the isomorphisms $Q_0 z \cong Q_u \times M_0 z$ and $P_0 z \cong Q_u \times (P \cap M_0)z$. From the above step and since $X^\circ = P_u Z_0 \cong P_u \times Z_0$, for a sufficiently general point $z \in Z_0$ we have $\overline{Q_0 z} = \overline{P_0 z}$. Thus for a sufficiently general point $z \in Z_0$ the previous equalities give

$$\overline{(P_u \cap M)z} = \overline{M_0 z}.$$

Step 3. Let us prove that $M_0 z \cong M_0/P_{M_0}^- \cong M/P_M^-$ for $z \in Z_0$, where $P_{M_0}^- := P^- \cap M_0$.

Step 3a). First assume that $z \in Z_0$ is a sufficiently general point. Let us notice that $(P_u \cap M)z$ maps isomorphically to $(P_u \cap M)P_M^-/P_M^-$. The M-equivariance of π_D and the fact that π_D is an isomorphism on the dense subset $(P_u \cap M_0)z$ in $M_0 z$ imply that π_D maps $M_0 z$ isomorphically to M/P_M^-. In addition we have $(M_0)_z = (M_0)_{\pi_D(z)} = P_{M_0}^-$.

Step 3b). Consider an arbitrary point $z \in Z_0$. Since the set of $P_{M_0}^-$-fixed points is dense (by Step 3a) and closed in Z_0, we get the inclusion $(M_0)_z \supset P_{M_0}^-$. Using the M-equivariance of π_D we get the following sequence of inclusions:

$$P_{M_0}^- \subset (M_0)_z \subset (M_0)_{\pi_D(z)} = P_{M_0}^-.$$

The right and left sides of this chain are equal, thus all the inclusions must be equalities, in particular $(M_0)_z = P_{M_0}^-$ for all $z \in Z_0$.

Since $Z_1 \cong M *_{P_M^-} Z_0$ and $P_{M_0}^-$ acts on Z_0 trivially we get $Z_1 \cong M/P_M^- \times Z_0$, where M acts on M/P_M^- by left multiplication and via the quotient $A = M/M_0 = P_M^-/P_{M_0}^-$ on Z_0. ■

To get the variant of the Local Structure Theorem obtained by Timashev [18, Thm. 3], we have first to take an open subset of $Z_0 = \pi_D^{-1}\pi_D(x_0) \cap \psi_E^{-1}\psi_E(x_0)$ on which the A-action is free. Taking a smaller open subset of Z_0, we may suppose that it is isomorphic to $A \times C$, where the action on C is trivial.

Corollary 4.3. *(cf. [18, Thm. 3]) There is a locally closed subset C of X such that the map*

$$Q *_M (M/P_M^- \times A \times C) \longrightarrow X$$

is an isomorphism to an open Q-invariant subset of X.

Remark 4.4. In Theorem 4.1 we can take any parabolic subgroup Q which is the stabilizer of some B-invariant divisor E and which satisfies the condition $T\cap[M,M]\subset L_0$. Corollary 3.5 states that such a parabolic subgroup Q stabilizes the divisors of all B-semi-invariant rational functions on X and the proof of Theorem 4.1 can be generalized to the case of such Q. However, instead of Step 1 of the proof of Theorem 4.1, one can use the argument from Timashev [18, Claims 1,2] to prove that $\mathfrak{m}_0 x\subseteq \mathfrak{p}_u x$ for general $x\in X^\circ$. In particular, that will imply that $\mathfrak{q}_0 x=\mathfrak{p}_u x$ as well as that $P_u x$ is open in $Q_0 x$.

5. Families of nongeneric horospheres

In this section we shall construct a family of nongeneric horospheres. By horospheres we mean the orbits of maximal unipotent subgroups of G. It will be proved that for the conormal bundle $\mathcal{N}^*_X$ to some foliation of $\overline{U}$-orbits (for some maximal unipotent subgroup $\overline{U}$) constructed below, we have $\overline{G\mathcal{N}^*_X}=T^*_X$. The construction is based on ideas of Knop [12]. Our main idea is to construct a Białynicki-Birula cell by means of special choice of a one-parameter subgroup that allows us to avoid using compactifications as in the cited paper of Knop. It also provides a deeper study of the constructed conormal bundle. This section is independent of the previous ones, so some notation will be slightly changed for brevity.

The crucial step is first to consider the case of a horospherical variety. Let us recall that a variety is called horospherical if the stabilizer of a general point contains a maximal unipotent subgroup. The study of the case of a general variety X will be reduced to the study of the case of a horospherical variety by means of so-called horospherical contraction, whose definition and existence is stated below in Proposition 5.10. In a horospherical variety X one can find a G-invariant open subset isomorphic to $G/P_0^-\times C$, where C is supplied with the trivial action of G and $P_0^-=L_0P_u^-$. Thus we shall construct a variety of degenerate horospheres for $X=G/P_0^-$ and extend it to the horospherical variety X by taking the product with C. Let us introduce the additional notation $M:=Z_G(\mathfrak{a})$ and $M_0:=[M,M]Z(L_0)$. We note that M is not related to the group introduced in Section 2 and this notation retains to the end of the paper.

Proposition 5.1. *Let X be the horospherical variety G/P_0^-. Consider a Borel subgroup $\overline{B}\subset G$ such that $\overline{\mathfrak{b}}$ contains the solvable subalgebra $\mathfrak{a}+(\mathfrak{p}_u^-\cap\mathfrak{m})$, and let us denote by $\overline{U}$ its unipotent radical. Let $\mathcal{N}^*_X$ be the conormal bundle to the orbit $\overline{U}P_0^-/P_0^-$, then we have $\overline{G\mathcal{N}^*_X}=T^*_X$.*

Proof. The cotangent bundle T^*_X is identified with $G*_{P_0^-}\mathfrak{p}_0^{-\perp}\cong G*_{P_0^-}(\mathfrak{a}+\mathfrak{p}_u^-)$. The fiber of the conormal bundle to the orbit $\overline{U}P_0^-/P_0^-$ at the point eP_0^- is identified with $(\overline{\mathfrak{u}}+\mathfrak{p}_0^-)^\perp=\overline{\mathfrak{u}}^\perp\cap\mathfrak{p}_0^{-\perp}=\overline{\mathfrak{b}}\cap(\mathfrak{a}+\mathfrak{p}_u^-)\supset\mathfrak{a}+(\mathfrak{p}_u^-\cap\mathfrak{m})$.

We need the following lemma.

Lemma 5.2. *Let P be some parabolic subgroup of G and L be a Levi subgroup* *of P. Then for any subalgebra $\mathfrak{a} \subset \mathfrak{z}(\mathfrak{l})$ we have $\overline{P_u(\mathfrak{a} + (\mathfrak{p}_u \cap \mathfrak{m}))} = \mathfrak{a} + \mathfrak{p}_u$.*

Proof. We shall prove that the map $P_u \times (\mathfrak{a} + (\mathfrak{p}_u \cap \mathfrak{m})) \to \mathfrak{a} + \mathfrak{p}_u$ is dominant by proving that its differential is surjective at the point (e, ξ), for general $\xi \in \mathfrak{a}$. Calculating the differential in (e, ξ) and using the equality $\mathfrak{p}_u = (\mathfrak{p}_u \cap \mathfrak{z}_{\mathfrak{g}}(\xi)) \oplus [\mathfrak{p}_u, \xi]$ for any $\xi \in \mathfrak{a}$ and the equality $\mathfrak{p}_u \cap \mathfrak{z}_{\mathfrak{g}}(\xi) = \mathfrak{p}_u \cap \mathfrak{m}$ which holds for general $\xi \in \mathfrak{a}$, we obtain that the differential:

$$\mathfrak{p}_u \times (\mathfrak{a} + (\mathfrak{p}_u \cap \mathfrak{m})) \to [\mathfrak{p}_u, \xi] + \mathfrak{a} + (\mathfrak{p}_u \cap \mathfrak{m}) = \mathfrak{a} + \mathfrak{p}_u$$

is indeed surjective. ∎

The group P_u^- acts on the fiber of T_X^* over $x_0 = eP_0^-$ since it lies in stabilizer of this point. From the preceding lemma we obtain that

$$\overline{P_u^- \mathcal{N}_{X,x_0}^*} \supset \overline{P_u^-(\mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m}))} = \mathfrak{a} + \mathfrak{p}_u^- = T_{X,x_0}^*.$$

This implies that

$$\overline{G\mathcal{N}_X^*} = G(T_{X,x_0}^*) = T_X^*.$$ ∎

We shall need the following elementary lemma.

Lemma 5.3. *Let X/S and Y/S be two families of equidimensional varieties over some variety S with an S-morphism $f\colon X/S \to Y/S$. Suppose there exist smooth points $s_0 \in S$ and $x_0 \in X_{s_0}$ such that the varieties X, X_{s_0} are smooth at $x_0 \in X_{s_0}$, the varieties Y, Y_{s_0} are smooth at $f(x_0)$, the map $f_{s_0}\colon X_{s_0} \to Y_{s_0}$ is a submersion at x_0 (that is the map of tangent spaces $df_{s_0}\colon T_{X_{s_0},x_0} \to T_{Y_{s_0},f(x_0)}$ is surjective), and the projections $X \to S$, $Y \to S$ are submersions at x_0 and $f_{s_0}(x_0)$. Then the morphism $f_s\colon X_s \to Y_s$ is a submersion (and in particular is dominant) at a general point of X_s for a sufficiently general s.*

Before stating one of the main theorems of this section we recall the following proposition on adjoint orbits. Let us temporally change our notations only for this proposition.

Proposition 5.4. *[4, Sec. 5.1, 5.5] Consider an arbitrary parabolic subgroup P in G, a Levi subgroup L and the unipotent radical P_u. Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent adjoint orbit of L in $\mathfrak{l}$. Let $x \in \mathfrak{z}(\mathfrak{l})$ be an arbitrary element of the center of $\mathfrak{l}$. There exists a unique G-orbit $\mathcal{O}_{\mathfrak{g}}$ meeting $x + \mathcal{O}_{\mathfrak{l}} + \mathfrak{p}_u$ in a dense open subset. The intersection $\mathcal{O}_{\mathfrak{g}} \cap (x + \mathcal{O}_{\mathfrak{l}} + \mathfrak{p}_u)$ is a single P-orbit. The following equality holds $\operatorname{codim}_{\mathfrak{g}} \mathcal{O}_{\mathfrak{g}} = \operatorname{codim}_{\mathfrak{l}} \mathcal{O}_{\mathfrak{l}}$. For a general point $z \in x + \mathcal{O}_{\mathfrak{l}} + \mathfrak{p}_u$ the stabilizer of z in $\mathfrak{p}_u^-$ is trivial and $[\mathfrak{p}_u^-, z]$ is transversal to $[\mathfrak{l}, z] + \mathfrak{p}_u$. We also have the equality of irreducible components $(P_z)^0 = (G_z)^0$.*

Now we are ready to deal with the case when X is an arbitrary G-variety. We are going to construct a family of $\overline{U}$-orbits for some maximal unipotent subgroup $\overline{U}$ such that the G-translate of the conormal bundle to this foliation is dense in T_X^*.

Theorem 5.5. *Let X be a smooth G-variety. Consider the open subset $X^\circ \cong P *_L Z$ obtained by application of the Local Structure Theorem 1.6 to some effective B-divisor with stabilizer equal to the parabolic subgroup $P := P(X)$. Then there exists a maximal unipotent subgroup $\overline{U}$ with the following properties:*

(i) For any $z \in Z$ we have $\overline{U}z = (\overline{U} \cap U)z$.

(ii) Let $\mathcal{N}^*_X$ be the conormal bundle to the foliation of orbits $\overline{U}z$ for $z \in Z$. Then we have $\overline{G\mathcal{N}^*_X} = T^*_X$.

Proof. To construct the desired family we proceed in several steps.
Step 1. Our aim is to construct a Białynicki-Birula cell with respect to a one-parameter subgroup $\lambda \in \Lambda(Z(L_0))$. We shall choose λ in a special way. Let us recall that $\mathfrak{p} \cap \mathfrak{m}_0$ is a parabolic subalgebra of $\mathfrak{m}_0$ with Levi subalgebra $\mathfrak{l}_0$ and nilpotent radical $\mathfrak{p}_u \cap \mathfrak{m}$.

Let us take a one-parameter subgroup $\lambda\colon \mathbb{K}^\times \to T$ such that $\lambda(t) \in Z_{M_0}(L_0)$ and $\langle \lambda, \gamma \rangle < 0$ for all $\gamma \in \Delta_{\mathfrak{p}_u \cap \mathfrak{m}}$.

Let us introduce the following groups: We put $\overline{M} := Z_G(\lambda)$, with root system $\Delta_{\overline{M}} = \{\gamma \in \Delta | \langle \gamma; \lambda \rangle = 0\}$. Then $\overline{M}$ is a Levi subgroup of

$$\overline{Q} := \{g \in G | \text{ there exists } \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ in } G\},$$

with Lie algebra

$$\overline{\mathfrak{q}} = \mathfrak{t} \oplus \bigoplus_{\langle \alpha, \lambda \rangle \geqslant 0} \mathfrak{g}_\alpha.$$

The unipotent radical of $\overline{Q}$ and the corresponding Lie algebra can be expressed by the formulae

$$\overline{Q}_u = \{g \in G | \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} = e\}, \qquad \overline{\mathfrak{q}}_u = \bigoplus_{\langle \alpha, \lambda \rangle > 0} \mathfrak{g}_\alpha.$$

In particular, we have the following obvious inclusions: $\overline{M} \supset L$ and $\overline{\mathfrak{q}}_u \supset \mathfrak{p}_u^- \cap \mathfrak{m}$.

Let us fix an open P-invariant subset $X^\circ = P *_L Z$ of X constructed in the Local Structure Theorem 1.6 applied to some effective B-divisor with stabilizer equal to the parabolic subgroup $P := P(X)$. We recall that L_0 acts trivially on Z. Consider the following open subset of a Białynicki-Birula cell:

$$Z_\lambda := \{x \in X | \lim_{t\to 0} \lambda(t) x \in Z\}.$$

It is well defined since $\lambda(t)$ fixes the points of Z. Let us define a map φ by the formula:

$$\varphi\colon Z_\lambda \to Z, \quad \varphi(x) = \lim_{t\to 0} \lambda(t) x.$$

Let us show that $Z_\lambda \subset X^\circ$. Indeed, if $\lim_{t\to 0} \lambda(t)x = z \in Z$, then the λ-orbit of x intersects X°, which is an open neighborhood of z. Thus, being λ-invariant X° contains the whole λ-orbit of x.

Lemma 5.6. *For $x \in Z_\lambda$, $q \in \overline{Q}_u$ and $m \in \overline{M}$ we have the following equalities: $\varphi(qx) = \varphi(x)$ and $\varphi(mx) = m\varphi(x)$.*

Proof. Indeed, for $q \in \overline{Q}_u$ by definition $\lim_{t\to 0} \lambda(t)q\lambda(t)^{-1} = e$. Thus we get:

$$\varphi(qx) = \lim_{t\to 0} \lambda(t)q\lambda(t)^{-1} \cdot \lim_{t\to 0} \lambda(t)x = \lim_{t\to 0} \lambda(t)x = \varphi(x),$$

$$\varphi(mx) = \lim_{t\to 0} \lambda(t)mx = m \lim_{t\to 0} \lambda(t)x = \varphi(x). \qquad \blacksquare$$

Proposition 5.7. *For $z \in Z$ we have*
$$\varphi^{-1}(z) = (P_u \cap \overline{Q}_u)z \text{ and } Z_\lambda = (P_u \cap \overline{Q}_u)Z.$$

Proof. As was noticed before $Z_\lambda \subset X^\circ$. Let us write down the action of λ on a point $x \in X^\circ$, which we write in a form $x = p_u z$, for $p_u \in P_u$ and $z \in Z$. Since the action of λ is trivial on Z we get

$$\lim_{t\to 0} \lambda(t)(p_u z) = \lim_{t\to 0} \lambda(t)p_u\lambda(t)^{-1}z.$$

Taking into account that $\lambda(t)p_u\lambda(t)^{-1} \in P_u$ and the fact that the action of P_u is free on X° we get that $\lim_{t\to 0} \lambda(t)p_u\lambda(t)^{-1}z$ exists iff $\lim_{t\to 0} \lambda(t)p_u\lambda(t)^{-1}$ exists. This implies that $p_u \in \overline{Q}$.

Let us prove that $p_u \in \overline{Q}_u$. Using the Levi decomposition $\overline{Q} = \overline{M} \ltimes \overline{Q}_u$ we get the decomposition $P_u \cap \overline{Q} = P_u \cap \overline{M} \ltimes P_u \cap \overline{Q}_u$, and in particular $p_u = mq_u$ for $m \in P_u \cap \overline{M}$ and $q_u \in P_u \cap \overline{Q}_u$. Hence we obtain

$$\lim_{t\to 0} \lambda(t)(p_u z) = m \lim_{t\to 0} \lambda(t)(q_u z) = mz.$$

Thus the inclusion $mz \in Z$ is satisfied if and only if $m = e$, since $m \in P_u$, $X^\circ \cong P_u \times Z$. This gives the desired inclusion $p_u \in \overline{Q}_u$ and completes the proof. $\blacksquare$

Consider the $\overline{Q}_u$-orbits of the points from Z. We shall prove that these orbits are contained in the open subset X°.

Lemma 5.8. *For $z \in Z$ we have $(P_u \cap \overline{Q}_u)z = \overline{Q}_u z$.*

Proof. By Lemma 5.6 and Proposition 5.7 we get $\overline{Q}_u z \subseteq \varphi^{-1}(z) = (P_u \cap \overline{Q}_u)z$. That implies our lemma. $\blacksquare$

Step 2. Let us define the group $\overline{U}$. Consider the group $\overline{U}_{\overline{M}} = U \cap \overline{M}$. Set $\overline{U} := \overline{U}_{\overline{M}} \ltimes \overline{Q}_u \subset \overline{Q}$. Being the preimage of a maximal unipotent subgroup in $\overline{M}$ under the morphism $\overline{Q} \to \overline{Q}/\overline{Q}_u \cong \overline{M}$, the group $\overline{U}$ is a maximal unipotent subgroup of G.

Consider the family of orbits $\overline{U}z$ for $z \in Z$. From Lemma 5.8 we obtain that $\overline{U}z = \overline{U}_{\overline{M}}(\overline{Q}_u z) = \overline{U}_{\overline{M}}(\overline{Q}_u \cap P_u)z \subset P_u z$. This implies in particular that the orbits $\overline{U}z$ are contained in X° and that $\overline{U}z_1 \neq \overline{U}z_2$ for $z_1 \neq z_2$ (since $P_u z_1 \cap P_u z_2 = \varnothing$).

Lemma 5.9. *The orbit $\overline{U}z$ of $z \in Z$ is stable under the group*

$$\overline{S} := (L_0 \ltimes (\overline{M} \cap P_u)) \ltimes \overline{Q}_u \subset \overline{Q}.$$

Proof. We need to prove that L_0 normalizes $\overline{U}z$. But this follows from

$$L_0\overline{U}z = L_0(\overline{M} \cap P_u)\overline{Q}_u z = (\overline{M} \cap P_u)\overline{Q}_u L_0 z = \overline{U}z,$$

where we have used that $L_0 \subset G_z$. ■

Now we are ready to prove part (ii) of the Theorem 5.5. We shall give two proofs: one is based on degeneration to horospherical variety, the other is based on the calculation of the image of the moment map[2]. Let us recall the definition of the conormal bundle to the foliation of $\overline{U}$-orbits:

$$\mathcal{N}_X^* = \{\xi \in T_{X,x}^* |\ x \in \overline{U}Z, \quad \langle \overline{\mathfrak{u}}x, \xi \rangle = 0\}.$$

Proof. [First proof of Theorem 5.5 (ii).] By [9] (see also [16]) we know that every G-variety X admits a degeneration to a horospherical variety.

Proposition 5.10. *For a G-variety X there exists a $G \times \mathbb{K}^\times$-variety $\mathcal{X}$ and a surjective G-invariant morphism $\tau : \mathcal{X} \to \mathbb{A}^1$ that is equivariant with respect to the action $\mathbb{K}^\times : \mathbb{A}^1$, such that:*

(i) For $t \neq 0$ the fiber $X_t := \tau^{-1}(t)$ is isomorphic to X. The fiber X_0 is a smooth horospherical variety. Shrinking X we may assume that $X_0 \cong G/P_0^- \times C$ (for some variety C).

(ii) The morphism τ is equidimensional and flat. By shrinking X and $\mathcal{X}$, we can assume that $\mathcal{X}$ and τ are smooth.

(iii) For the fibers of τ we have $P(X_t) = P$, the group L_0 is independent of t and in particular $\mathfrak{a}_{X_t} = \mathfrak{a}$.

Let us choose a B-invariant divisor $D \subset X$ with stabilizer $P := P(X)$. We extend this divisor to a $B \times \mathbb{K}^\times$–divisor $\mathcal{D}$ on the $G \times \mathbb{K}^\times$–variety $\mathcal{X}$ in the following way. Using the isomorphism $\mathcal{X} \setminus X_0 \cong X \times \mathbb{K}^\times$ we extend D to a $B \times \mathbb{K}^\times$–divisor on $\mathcal{X} \setminus X_0$. We are finished by setting $\mathcal{D}$ to be the closure of this divisor.

Having constructed the $B \times \mathbb{K}^\times$–divisor $\mathcal{D}$, we see that the $P[D]$-regularity condition for the weight χ_D of the section σ_D is the same as the $P[\mathcal{D}]$-regularity condition of the weight $\chi_{\mathcal{D}}$ of the section $\sigma_{\mathcal{D}}$. So we can apply the Local Structure Theorem to get the following proposition.

Proposition 5.11. *Consider the map $\psi_{\mathcal{D}} : \mathcal{X} \setminus \mathcal{D} \to \mathfrak{g}^*$ constructed in the Local Structure Theorem. Then $\operatorname{Im} \psi_{\mathcal{D}} = \chi_D + \mathfrak{p}_u$. Defining $\mathcal{Z} := \psi_{\mathcal{D}}^{-1}(\chi_D)$, we have*

$$\mathcal{X} \setminus \mathcal{D} \cong P *_L \mathcal{Z}.$$

If X is an affine variety, $D = (f)$ for some $f \in \mathbb{K}[X]^{(B)}$ and $\mathcal{X}$ is a contraction to the affine horospherical variety in the sense of [16], then $\mathcal{Z} \cong Z \times \mathbb{A}^1$.

[2] It is a generalization of Knop's proof [10] for nondegenerate varieties.

Proof. Let X be an affine variety. We are left to prove that $\mathcal{Z} \cong Z \times \mathbb{A}^1$. Let us recall that we have a $\mathbb{K}^\times$-equivariant isomorphism $\mathcal{X}/\!\!/U \cong X/\!\!/U \times \mathbb{A}^1$ ([16, Prop. 11]). Let $F \in \mathbb{K}[\mathcal{X}]^{(B)}$ be the pullback of f under the projection $\mathcal{X}/\!\!/U \to X/\!\!/U$. Since the divisor of zeroes of $F|_{X_1} = f$ is equal to D and F does not vanish identically on X_0, the $\mathbb{K}^\times$-invariance of $\mathcal{D}$ and F implies $\mathcal{D} = (F)$. The desired equality follows from the chain

$$\mathbb{K}[\mathcal{Z}] = \mathbb{K}[\mathcal{X} \setminus \mathcal{D}]^U = K[\mathcal{X}]^U_F = K[X]^U_f \otimes \mathbb{K}[t] = K[Z] \otimes \mathbb{K}[t],$$

where $K[\mathcal{X}]^U_F, K[X]^U_f$ are the localizations with respect to F and f respectively. ■

The family of orbits constructed in Step 2 can be extended to the whole variety $\mathcal{X}$. Since $P(X) = P(X_0)$ and $\mathfrak{a}_X = \mathfrak{a}_{X_0}$ we can choose the same λ as in Step 1 for the variety $\mathcal{X}$ and all its fibers X_t. We define a Białynicki-Birula cell for $\mathcal{X}$ as follows

$$\mathcal{Z}_\lambda = \{x \in \mathcal{X} |\ \exists \lim_{t\to 0} \lambda(t)x \in \mathcal{Z}\}.$$

Applying Proposition 5.7 we get

$$\mathcal{Z}_\lambda = (P_u \cap \overline{Q}_u) \times \mathcal{Z}.$$

Consider the conormal bundle $\mathcal{N}^*_X$ to the constructed foliation of $\overline{U}$-orbits. It fits into the following family of the conormal bundles to the foliations of $\overline{U}$-orbits in the fibers of $\tau\colon \mathcal{X} \to \mathbb{A}^1$:

$$\mathcal{N}^*_{\mathcal{X}/\mathbb{A}^1} = \{\xi \in T^*_{\mathcal{X},x}/\tau^*(T^*_{\mathbb{A}^1,x}) |\ x \in \overline{U}\mathcal{Z}_t, \quad \langle \overline{\mathfrak{u}}x, \xi\rangle = 0\}.$$

We note that the restrictions of $T^*_{\mathcal{X}}/\tau^*(T^*_{\mathbb{A}^1})$ and $\mathcal{N}^*_{\mathcal{X}/\mathbb{A}^1}$ to X_t are isomorphic to $T^*_{X_t}$ and $\mathcal{N}^*_{X_t}$. From Proposition 5.1 we know that the map $G \times \mathcal{N}^*_{X_0} \to T^*_{X_0}$ is dominant. This implies (by application of Lemma 5.3) that $G \times \mathcal{N}^*_{X_t} \to T^*_{X_t}$ is dominant for general t, which proves our claim. ■

[Second proof of Theorem 5.5 (ii).] The second proof is based on the study of the image of the conormal bundle $\mathcal{N}^*_X$ under the moment map μ_X. Since $\overline{S}$ normalizes the orbits $\overline{U}z$ (for $z \in Z$), we get

$$\mathcal{N}^*_X = \{\xi \in T^*_X |\ x \in \overline{U}Z, \quad \langle \overline{\mathfrak{s}}x, \xi\rangle = 0\},$$

$$\mu_X(\mathcal{N}^*_X) \subset \overline{\mathfrak{s}}^\perp = \mathfrak{a} + (\mathfrak{p}_u \cap \overline{\mathfrak{m}}) + \overline{\mathfrak{q}}_u \supset \mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m}).$$

Remark 5.12. By the construction of $\overline{\mathfrak{q}}$ we have the inclusion $\overline{\mathfrak{q}}_u \supset (\mathfrak{p}_u^- \cap \mathfrak{m})$ and the equality $\mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m}) = \mathfrak{a} + (\mathfrak{p}_u \cap \overline{\mathfrak{m}} + \overline{\mathfrak{q}}_u) \cap \mathfrak{m}$.

Let us denote $\overline{P} := N_G(\overline{S})$. We have $\overline{\mathfrak{p}}_u = \mathfrak{p}_u \cap \overline{\mathfrak{m}} + \overline{\mathfrak{q}}_u$. The next proposition gives us information about the image $\mu_X(\mathcal{N}^*_{X,z})$.

Proposition 5.13. *Let $\mathcal{N}^*_{X,z}$ be the fiber of $\mathcal{N}^*_X$ over $z \in Z$. Consider the T-equivariant projection of $\mathfrak{a} + \overline{\mathfrak{p}}_u$ to the subspace $\mathfrak{a} + \overline{\mathfrak{q}}_u \cap \mathfrak{p}_u^-$ with the fibers parallel to the subspace $\overline{\mathfrak{q}} \cap \mathfrak{p}_u$. Then the image of $\mu_X(\mathcal{N}^*_{X,z})$ under this projection is equal to $\mathfrak{a} + \overline{\mathfrak{q}}_u \cap \mathfrak{p}_u^-$.*

Proof. Consider the T-stable decomposition $\mathfrak{g} = \mathfrak{a} + (\overline{\mathfrak{q}}_u \cap \mathfrak{p}_u^- + \overline{\mathfrak{q}}_u^- \cap \mathfrak{p}_u) + \mathfrak{g}_0$, where $\mathfrak{g}_0$ is orthogonal to the other direct summands. The restriction of the pairing to $\overline{\mathfrak{q}}_u \cap \mathfrak{p}_u^- + \overline{\mathfrak{q}}_u^- \cap \mathfrak{p}_u$ is non-degenerate and the subspaces $\overline{\mathfrak{q}}_u \cap \mathfrak{p}_u^-$, $\overline{\mathfrak{q}}_u^- \cap \mathfrak{p}_u$ are isotropic. Moreover $\overline{\mathfrak{q}} \cap \mathfrak{p}_u \subset \mathfrak{g}_0$, and the elements of $\mathfrak{a} + (\overline{\mathfrak{q}}_u \cap \mathfrak{p}_u^-)$ are identified with the linear functions on $\mathfrak{a} + (\overline{\mathfrak{q}}_u^- \cap \mathfrak{p}_u)$. From the inclusion

$$T_{X,z} \supset \overline{\mathfrak{u}}z \oplus (\mathfrak{a} + \overline{\mathfrak{q}}_u^- \cap \mathfrak{p}_u)z = (\overline{\mathfrak{q}} \cap \mathfrak{p}_u)z \oplus (\mathfrak{a} + \overline{\mathfrak{q}}_u^- \cap \mathfrak{p}_u)z,$$

we see that any linear function η on $\mathfrak{a} + (\overline{\mathfrak{q}}_u^- \cap \mathfrak{p}_u)$ can be lifted to an element $\xi \in T^*_{X,z}$ that is zero on $(\overline{\mathfrak{q}} \cap \mathfrak{p}_u)z = \overline{\mathfrak{u}}z$. We found $\xi \in \mathcal{N}^*_{X,z}$ such that the projection of $\mu_X(\xi)$ to $\mathfrak{a} + \overline{\mathfrak{q}}_u \cap \mathfrak{p}_u^-$ is equal to η. This proves the proposition. ■

Remark 5.14. Consider some point $z \in Z$. The one-parameter subgroup λ acts on $T_{X,z}$ since z is fixed by λ. In the proof of Proposition 5.13 we constructed a subspace $V_z \subset \mathcal{N}^*_{X,z}$ such that $\mu_X(V_z)$ maps isomorphically to $\mathfrak{a} + \overline{\mathfrak{q}}_u \cap \mathfrak{p}_u^-$ under the T-equivariant projection. Let us notice that the decomposition

$$T_{X,z} = (\overline{\mathfrak{q}} \cap \mathfrak{p}_u)z \oplus (\mathfrak{a} + \overline{\mathfrak{q}}_u^- \cap \mathfrak{p}_u)z \oplus R$$

can be taken λ-equivariant. This implies that V_z can be chosen λ-invariant.

Proposition 5.15. *We have the equality* $\overline{\mu_X(\mathcal{N}^*_X)} = \mathfrak{a} + \overline{\mathfrak{p}}_u = \mathfrak{a} + \mathfrak{p}_u \cap \overline{\mathfrak{m}} + \overline{\mathfrak{q}}_u$.

We shall give three proofs of this proposition based on different ideas. The first proof is based on a transversality argument, the second proof is based on the construction of a family of linear subspaces that tends to $\mathfrak{a} + \overline{\mathfrak{q}}_u \cap \mathfrak{p}_u^-$ and the third proof is based on a degeneration to a horospherical variety.

Proof. [First proof of Proposition 5.15] Since $\overline{P}$ normalizes the foliation of the $\overline{U}$-orbits it also normalizes $\mathcal{N}^*_X$. Thus to prove the proposition it is sufficient to show that $\overline{P}\mu_X(\mathcal{N}^*_{X,z})$ is dense in $\overline{\mathfrak{s}}^{\perp} = \mathfrak{a} + \overline{\mathfrak{p}}_u$. We shall use the following lemma.

Lemma 5.16. *Consider the action of* $P_u \cap \overline{Q}$ *on* $\mathfrak{a} + \overline{\mathfrak{p}}_u$. *Let* $\xi \in \mathfrak{a} + \overline{\mathfrak{p}}_u$ *be a general point. Then the stabilizer of* ξ *in* $P_u \cap \overline{Q}$ *is trivial. Moreover, if* $\xi \in \mathfrak{a}^{pr} + \mathfrak{p}_u \cap \overline{\mathfrak{q}}$ *then* $(P_u \cap \overline{Q})\xi = \xi + \mathfrak{p}_u \cap \overline{\mathfrak{q}}$.

Proof. The first claim follows from the second one. Let us take $\xi \in \mathfrak{a}^{pr}$. We have the inclusion $(P_u \cap \overline{Q})\xi \subset \xi + \mathfrak{p}_u \cap \overline{\mathfrak{q}}$ (it follows from the equality $P_u \cap \overline{Q} = \exp(\mathfrak{p}_u \cap \overline{\mathfrak{q}})$ and the inclusion $[\xi, \mathfrak{p}_u \cap \overline{\mathfrak{q}}] \subset \mathfrak{p}_u \cap \overline{\mathfrak{q}}$). From $\mathfrak{p}_u \cap \mathfrak{z}_{\mathfrak{g}}(\xi) = \mathfrak{p}_u \cap \mathfrak{m} \subset \overline{\mathfrak{q}}_u^-$ we get that the stabilizer of ξ in $P_u \cap \overline{Q}$ is trivial and we have the equality $[\xi, \mathfrak{p}_u \cap \overline{\mathfrak{q}}] = \mathfrak{p}_u \cap \overline{\mathfrak{q}}$. This implies that the tangent space in ξ to the orbit $(P_u \cap \overline{Q})\xi$ coincides with $\mathfrak{p}_u \cap \overline{\mathfrak{q}}$. Thus $(P_u \cap \overline{Q})\xi$ is dense in $\xi + \mathfrak{p}_u \cap \overline{\mathfrak{q}}$. Since any orbit of a unipotent group in an affine variety is closed we get $(P_u \cap \overline{Q})\xi = \xi + \mathfrak{p}_u \cap \overline{\mathfrak{q}}$. ■

By Proposition 5.13 we get that there exists $\xi = \xi_0 + \xi_+ \in \mu_X(\mathcal{N}^*_{X,z})$, where $\xi_0 \in \mathfrak{a}^{pr}$ and $\xi_+ \in \mathfrak{p}_u \cap \overline{\mathfrak{q}}$. By Lemma 5.16 we get that the stabilizer of ξ in $\mathfrak{p}_u \cap \overline{\mathfrak{q}}$ is trivial and $[\mathfrak{p}_u \cap \overline{\mathfrak{q}}, \xi] = \mathfrak{p}_u \cap \overline{\mathfrak{q}}$. Since the projection of $\mu_X(\mathcal{N}^*_{X,z})$ to the subspace $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ is surjective we get that $\mu_X(\mathcal{N}^*_{X,z}) + \mathfrak{p}_u \cap \overline{\mathfrak{q}} = \overline{\mathfrak{s}}^{\perp}$. Calculating the differential of the map $(P_u \cap \overline{Q}) \times \mu_X(\mathcal{N}^*_{X,z}) \to (P_u \cap \overline{Q})\mu_X(\mathcal{N}^*_{X,z})$ at the point ξ we get

$$(\mathfrak{p}_u \cap \overline{\mathfrak{q}}) \times \mu_X(\mathcal{N}^*_{X,z}) \to [\mathfrak{p}_u \cap \overline{\mathfrak{q}}, \xi] + \mu_X(\mathcal{N}^*_{X,z}) = \overline{\mathfrak{s}}^{\perp}.$$

This implies that $(P_u \cap \overline{Q})\mu_X(\mathcal{N}^*_{X,z})$ is dense in $\overline{\mathfrak{s}}^{\perp}$, which proves the proposition. ∎

Proof. [Second proof of Proposition 5.15] Assume for a moment that we have constructed a one-parameter family of linear subspaces $V_t \subset \overline{\mathfrak{s}}^{\perp}$ such that $V_t \subset \mu_X(\mathcal{N}^*_{X,z_t})$ for some $z_t \in Z$, for any $t \neq 0$, and $V_0 = \mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$. The set $\overline{P}_u V_0$ contains $\overline{P}_u(\mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m}))$, which is dense in $\mathfrak{a} + \overline{\mathfrak{p}}_u = \overline{\mathfrak{s}}^{\perp}$ by Lemma 5.2. Applying Lemma 5.3 to the map $f_t \colon \overline{P_u} \times V_t \to \overline{\mathfrak{s}}^{\perp}$ and using that f_0 is dominant, we get that f_t is dominant for almost all t. The fact that $\overline{P}_u V_t$ is dense in $\overline{\mathfrak{s}}^{\perp}$ and the following chain of inclusions proves the proposition:

$$\mu_X(\mathcal{N}^*_X) \supset \overline{P}_u \mu_X(\mathcal{N}^*_{X,z_t}) \supset \overline{P}_u V_t.$$

To construct the desired family of subspaces let us fix a strictly dominant one-parameter subgroup $\lambda_0 \colon \mathbb{K}^* \to T$. By Proposition 5.13 the projection of $\mu_X(\mathcal{N}^*_{X,z})$ for $z \in Z$ to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ is surjective, thus we can fix a subspace $V_1 \subset \mu_X(\mathcal{N}^*_{X,z})$ that maps isomorphically to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$.

For the desired family let us take the closure of the family $V_t := \lambda_0(t)V_1 \subset \mu_X(\mathcal{N}^*_{X,\lambda_0(t)z})$ in the Grassmannian of subspaces of dimension $\dim(V_1)$ of $\mathfrak{g}$. By the next lemma (that is the characterization of the open Schubert cell as a Białynicki-Birula cell) we get that in this Grassmannian we have the following limit: $\langle V_0 \rangle := \lim_{t\to 0}\langle V_t \rangle = \langle \mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u \rangle$.

Lemma 5.17. *[17, Prop. 8.5.1] Let $\lambda(t)$ be a one-dimensional torus acting on a linear space V. Let $V_{\leqslant 0}$ be the sum of components of V with nonpositive λ-weights, and $\pi_{\leqslant 0}$ be the corresponding λ-equivariant projection. Consider a subspace $W \subset V$ that maps isomorphically to $V_{\leqslant 0}$ under the projection $\pi_{\leqslant 0}$. Then we have the following limit in the Grassmannian:*

$$\lim_{t\to 0} \lambda(t)\langle W \rangle = \langle V_{\leqslant 0} \rangle. \qquad ∎$$

Proof. [Third proof of Proposition 5.15] Let us include $\mathcal{N}^*_X$ into the family $\mathcal{N}^*_{\mathcal{X}/\mathbb{A}^1}$ as we have done in Argument 1. We also know that $\mu_{X_t}(\mathcal{N}^*_{X_t}) \subset \mathfrak{a} + \overline{\mathfrak{p}}_u$. We can do explicit calculations of the moment map for horospherical variety $X_0 \cong G/P_0^- \times C$. Indeed, the image $\mu_{X_0}(\mathcal{N}^*_{X_0,x_0})$ for $x_0 = eP_0^-$ is identified with

$$(\overline{\mathfrak{u}} + \mathfrak{p}_0^-)^{\perp} = \overline{\mathfrak{b}} \cap (\mathfrak{a} + \mathfrak{p}_u^-) = \mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u \supset \mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m}) = \mathfrak{a} + (\overline{\mathfrak{p}}_u \cap \mathfrak{m}).$$

For a horospherical variety X_0 we have

$$\mu_{X_0}(\mathcal{N}^*_{X_0}) = \overline{P}_u \mu_{X_0}(\mathcal{N}^*_{X_0,x_0}) \supset \overline{P}_u(\mathfrak{a} + (\overline{\mathfrak{p}}_u \cap \mathfrak{m}))$$

The above chain of inclusions and the density of $\overline{P}_u(\mathfrak{a} + (\overline{\mathfrak{p}}_u \cap \mathfrak{m}))$ in $\mathfrak{a} + \overline{\mathfrak{p}}_u$ (see Lemma 5.2) imply that $\overline{\mu_{X_0}(\mathcal{N}^*_{X_0})} = \mathfrak{a} + \overline{\mathfrak{p}}_u$.

Set $\mu_{\mathfrak{X}}(\alpha) := (\mu_{X_t}(\alpha), t) \in (\mathfrak{a} + \overline{\mathfrak{p}}_u) \times \mathbb{A}^1$ for $\alpha \in \mathcal{N}^*_{X_t}$. Let us apply Lemma 5.3 to the triple of varieties $(\mathcal{N}^*_{\mathfrak{X}/\mathbb{A}^1}, (\mathfrak{a} + \overline{\mathfrak{p}}_u) \times \mathbb{A}^1, \mathbb{A}^1)$ and the map $\mu_{\mathfrak{X}}$. Using the equality $\overline{\mu_{X_0}(\mathcal{N}^*_{X_0})} = \mathfrak{a} + \overline{\mathfrak{p}}_u$ we get that for a general $t \in \mathbb{A}^1$ we have $\overline{\mu_{X_t}(\mathcal{N}^*_{X_t})} = \mathfrak{a} + \overline{\mathfrak{p}}_u$. This proves the proposition. ■

Theorem 5.5 (ii) now follows from the next proposition.

Proposition 5.18. *The constructible set $\overline{P}_u^- \mathcal{N}^*_X$ is dense in T^*_X.*

Proof. By considering the differential of the map $\overline{P}_u^- \times \mathcal{N}^*_X \to \overline{P}_u^- \mathcal{N}^*_X$ we see that it is sufficient to prove that for some $\alpha \in \mathcal{N}^*_X$ the tangent space to T^*_X at the point α is equal to the sum of $\overline{\mathfrak{p}}_u^- \alpha$ and the tangent space to $\mathcal{N}^*_X$. Let us take α such that $\xi = \mu_X(\alpha) \in \mathfrak{a} + \overline{\mathfrak{p}}_u$ is a sufficiently general point. Proposition 5.4 implies that that $d\mu_X$ maps isomorphically $\overline{\mathfrak{p}}_u^- \alpha$ onto $[\overline{\mathfrak{p}}_u^-, \xi] \cong \overline{\mathfrak{p}}_u^-$ and that $[\overline{\mathfrak{p}}_u^-, \xi]$ is transversal to $\mathfrak{a} + \overline{\mathfrak{p}}_u = \overline{\mu_X(\mathcal{N}^*_X)}$ in the point ξ. This implies that $\overline{\mathfrak{p}}_u^- \alpha$ is transversal to $\mathcal{N}^*_X$ in α. The transversality of $\overline{\mathfrak{p}}_u^- \alpha$ and $\mathcal{N}^*_X$ combined with the equality $\operatorname{codim}_{T^*_X} \mathcal{N}^*_X = \dim P_u = \dim \overline{P}_u$ implies our claim. ■

The proof of Theorem 5.5 is completed. ■

We recover the following result of Knop as a corollary of Theorem 5.5.

Corollary 5.19. *([9], Thm. 5.4) The closure of the image of the moment map is equal to $\overline{\mu_X(T^*_X)} = G(\mathfrak{a} + \overline{\mathfrak{p}}_u) = G(\mathfrak{a} + \mathfrak{p}_u^-)$.*

Proof. We have $\overline{\mu_X(T^*_X)} = \overline{G\mu_X(\mathcal{N}^*_X)} = G(\mathfrak{a} + \overline{\mathfrak{p}}_u)$. The equality $G(\mathfrak{a} + \overline{\mathfrak{p}}_u) = G(\mathfrak{a} + \mathfrak{p}_u^-)$ follows from Lemma 5.2 and the equality $(\overline{\mathfrak{p}}_u \cap \mathfrak{m}) = (\mathfrak{p}_u^- \cap \mathfrak{m})$. ■

The following theorem describes the normalizer for the family of $\overline{U}$-orbits parameterized by Z.

Theorem 5.20. *The normalizer of the orbit $\overline{U}z$ for $z \in Z$ is equal to $\overline{S}$. The equality $g\overline{U}z = \overline{U}z'$ for some $z, z' \in Z$ holds iff $g \in \overline{P}$ (where $\overline{P} = N_G(\overline{S})$). The map $G *_{\overline{P}} \mathcal{N}^*_X \to T^*_X$ is dominant and generically finite.*

Proof. [3] For $z \in Z$ let $\widetilde{S}$ be the normalizer of $\overline{U}z$. Since $N_G(\widetilde{S})$ is a parabolic subgroup of G it also contains $N_G(\overline{U})$ which is a unique Borel subgroup containing $\overline{U}$; thus $T \subset N_G(\widetilde{S})$. Let us denote by $\widetilde{L}_0$ the Levi subgroup of $\widetilde{S}$ that contains L_0 and is normalized by T. Let $\widetilde{S}_z$ be the stabilizer in $\widetilde{S}$ of the point z, then we have $\widetilde{S} = \overline{U}\widetilde{S}_z$. Taking the quotient of this equality by the unipotent radical $\widetilde{S}_u$ of $\widetilde{S}$ (we note that $\widetilde{S}_u \subset \overline{U}$) we obtain that the image of the unipotent group $\overline{U}$ in $\widetilde{L}_0$ acts transitively on the homogeneous space $\widetilde{L}_0/\operatorname{Im}\widetilde{S}_z$ of the reductive group $\widetilde{L}_0$ (where $\operatorname{Im}\widetilde{S}_z$ denotes the image of $\widetilde{S}_z$ in $\widetilde{S}/\widetilde{S}_u$). From the following lemma we get that the homogeneous space $\widetilde{L}_0/\operatorname{Im}\widetilde{S}_z$ is a point and $\operatorname{Im}\widetilde{S}_z = \widetilde{L}_0$.

[3]The proof of the first assertion of Theorem 5.20 is similar to the proof [18, Sec.4 Lem.3]

Lemma 5.21. *A variety Y homogeneous under a reductive group G and an unipotent subgroup U of G is a point.*

Proof. Since Y is homogeneous under the unipotent group U it is affine and $\mathbb{K}[Y]^U = \mathbb{K}$. But since it is also homogeneous under the reductive group G we have a decomposition $K[Y] = \mathbb{K} \oplus M$ into a direct sum of G-modules. Since for the unipotent subgroup U the equality $M^U = 0$ imply that $M = 0$, we get that Y is a point. ■

From the Levi decomposition we get that $\widetilde{S}_z$ contains a subgroup conjugated to $\widetilde{L}_0$ by an element $s_u \in \widetilde{S}_u \subset \overline{U}$. Thus taking the point $s_u z$ instead of z we can assume that $\widetilde{S}_z$ contains $\widetilde{L}_0$. Since the action of P_u is free on X° the intersection $\widetilde{L}_0 \cap P_u \subset \widetilde{S}_z \cap P_u$ must be trivial, which implies $\widetilde{L}_0 = L_0$ and $\widetilde{S} = \overline{S}$.

Suppose that $g\overline{U}z = \overline{U}z'$ for some $z, z' \in Z$ and $g \in G$. The normalizers of the orbits $\overline{U}z$ and $\overline{U}z'$ are conjugate by g. Since they are both equal to $\overline{S}$ we have $g \in N_G(\overline{S}) = \overline{P}$. Conversely, $\overline{P}$ normalizes the family of orbits $\overline{U}z$ (i.e. the translate of $\overline{U}z$ by an element of $\overline{P}$ is equal to $\overline{U}z'$ for some $z' \in Z$).

It follows, in particular, that $\overline{P}$ normalizes $\mathcal{N}_X^*$. So the action map

$$G *_{\overline{P}} \mathcal{N}_X^* \to G\mathcal{N}_X^* \subset T_X^*$$

is dominant. Calculating the dimensions we get:

$$\dim \mathcal{N}_X^* = \dim Z + \dim \overline{U}z + \operatorname{codim}_X \overline{U}z = \dim T_X^* - \dim P_u,$$

$\dim \overline{P} = \dim P$ (since P and $\overline{P}$ have the common Levi subgroup L_0) and

$$\dim G *_{\overline{P}} \mathcal{N}_X^* = \dim G/\overline{P} + \dim \mathcal{N}_X^* = \dim T_X^*.$$

This implies that $G *_{\overline{P}} \mathcal{N}_X^* \to T_X^*$ is generically finite. ■

Remark 5.22. The proof of the theorem can be simplified if we assume that $z \in Z$ is sufficiently general. Let $\widetilde{S}$ be the normalizer of $\overline{U}z$, recall that $\widetilde{S}$ is normalized by T. Since $Z \cap \overline{U}z = z$ we have $\widetilde{S} \cap T = T_z = L_0 \cap T$. The number of horospherical subgroups H normalized by T and such that $H \cap T = T_0$ is finite, which shows that for a general $z \in Z$ the normalizer of $\overline{U}z$ is equal to some fixed subgroup $\widetilde{S}$. Let $\widetilde{P} \supset \overline{P}$ be the normalizer of the family of orbits $\overline{U}z$ for a general $z \in Z$, as above it is equal to $N_G(\widetilde{S})$. Clearly $\widetilde{P}$ acts on $\mathcal{N}_X^*$. Since $\dim \widetilde{P} \geqslant \dim \overline{P}$, calculating the dimensions as in the proof of Theorem 5.20 we get that the map $G *_{\overline{P}} \mathcal{N}_X^* \to T_X^*$ is dominant iff $\widetilde{P} = \overline{P}$.

6. Horospherical cotangent bundle

In this section we shall define a variety of degenerate horospheres $\mathcal{H}or_X$. Our aim is to prove that the conormal bundle to the family of degenerate horospheres maps birationally onto the cotangent bundle of $\mathcal{H}or_X$. This is a generalization of a theorem proved by Vinberg [19, Sec. 5, Thm. 3].

Consider the G-translates of horospheres from the foliation constructed in Theorem 5.5. By Theorem 5.20 we can identify this set with the variety

$\mathcal{H}or_X := G *_{\overline{P}} Z$, where $\overline{P}$ acts on Z via the quotient $A = \overline{P}/\overline{P}_0$. Since $\dim P = \dim \overline{P}$ we have $\dim \mathcal{H}or_X = \dim X$. Let us define the incidence variety:

$$\mathcal{U} := \{(x, \mathcal{H}) \in X \times \mathcal{H}or_X |\ x \in \mathcal{H}\}.$$

We note that a general point of X is contained in some $\mathcal{H} \in \mathcal{H}or_X$ (since GZ is dense in X). Thus the projection $p_X : \mathcal{U} \to X$ is dominant. The variety $\mathcal{U}$ can be identified with the subvariety $G *_{\overline{P}} \mathcal{U}_0$ of $G *_{\overline{P}} (X \times Z)$ (here $\overline{P}$ acts diagonally on $X \times Z$ via its standard action on X and via the quotient $\overline{P}/\overline{S}$ on Z) where

$$\mathcal{U}_0 = \{(x, z) \in X \times Z |\ x \in \overline{U}z\}.$$

Let us notice that Z can be diagonally embedded in $\mathcal{U}_0$.

For this incidence variety, following Vinberg we can define the skew conormal bundle, which we denote by HT^*_X (for details see [19, Sec. 4], [18, Sec. 2]).

The variety HT^*_X can be identified with the variety of triples $(x, \xi, \mathcal{H})$ such that

$$x \in \mathcal{H}, \quad \xi \in T^*_{X,x}, \quad \xi = 0 \text{ on } T_{\mathcal{H},x}.$$

From Theorem 5.20 we also get that HT^*_X is identified with $G *_{\overline{P}} \mathcal{N}^*_X$.

Consider the following commutative diagram:

$$\begin{array}{ccccc} T^*_X & \xleftarrow{\widehat{p}_X} & HT^*_X & \xrightarrow{\widehat{p}_{\mathcal{H}or_X}} & T^*_{\mathcal{H}or_X} \\ \downarrow & & \downarrow & & \downarrow \\ X & \xleftarrow{p_X} & \mathcal{U} & \xrightarrow{p_{\mathcal{H}or_X}} & \mathcal{H}or_X \end{array}$$

Theorem 5.20 can be restated in the following form:

Theorem 6.1. *The morphism* $HT^*_X \xrightarrow{\widehat{p}_X} T^*_X$ *is dominant and generically finite.*

We are ready to prove the following generalization of a result of Vinberg [19, Thm. 3].

Theorem 6.2. *The morphism* $HT^*_X \xrightarrow{\widehat{p}_{\mathcal{H}or_X}} T^*_{\mathcal{H}or_X}$ *is birational.*

Proof. Since $T^*_{\mathcal{H}or_X}$ is a vector bundle over $\mathcal{H}or_X$ and HT^*_X maps dominantly to $\mathcal{H}or_X$ it is sufficient to prove the claim of the theorem fiberwise. Let $\mathcal{H} \in \mathcal{H}or_X$, then the fiber of HT^*_X over $\mathcal{H}$ is identified with $N^*_{X/\mathcal{H}}$, the conormal bundle to $\mathcal{H} \subset X$. We shall prove that the image of $N^*_{X/\mathcal{H}}$ under $\widehat{p}_{\mathcal{H}or_X}$ contains an open subset of $T^*_{\mathcal{H}or_X,\mathcal{H}}$. Since all the maps are G-equivariant we can assume that $\mathcal{H} \in Z$, i.e. $\mathcal{H} = (P_u \cap \overline{Q})x$ for some $x \in Z$. We notice that the $P_u \cap \overline{Q}$-action on $N^*_{X/\mathcal{H}}$ is free and the fiber $\mathcal{N}^*_{X,x}$ of the conormal bundle to $\mathcal{H}$ at some point $x \in \mathcal{H}$ defines a section of this action. Without loss of generality we can shrink Z to get an open subset isomorphic to $A \times C$, so we can choose $x \in C$. By [19, Sec. 4] the morphism $\widehat{p}_{\mathcal{H}or_X}$ maps the fiber $\mathcal{N}^*_{X,x}$ isomorphically to the subspace $N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}} \subset T^*_{\mathcal{H}or_X,\mathcal{H}}$, where $\mathcal{H}or_{X,x} = p_X^{-1}(x)$ is the set of horospheres containing x, and $N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}}$ is the fiber over $\mathcal{H}$ of the conormal bundle to $\mathcal{H}or_{X,x}$ in $\mathcal{H}or_X$. (Since $p_{\mathcal{H}or_X}$ and p_X are both surjective and the dimension of the general fiber of $p_{\mathcal{H}or_X}$ is $\dim(\mathfrak{p}_u \cap \overline{\mathfrak{q}})$, we have $\dim \mathcal{H}or_{X,x} = \dim(\mathfrak{p}_u \cap \overline{\mathfrak{q}})$.) From the above it is clear that the birationality of $\widehat{p}_{\mathcal{H}or_X}$ will follow from the proposition below.

Proposition 6.3. *The $P_u \cap \overline{Q}$-action is generically free on $T^*_{\mathcal{H}or_X,\mathcal{H}}$. The subspace $N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}}$ intersects a general $P_u \cap \overline{Q}$-orbit from $T^*_{\mathcal{H}or_X,\mathcal{H}}$ transversally in a single point.*

Remark 6.4. The action of $P_u \cap \overline{Q}$ on $T^*_{\mathcal{H}or_X,\mathcal{H}}$ is well defined since $P_u \cap \overline{Q}$ stabilizes $\mathcal{H}$.

Proof. We note that the tangent space $T_{\mathcal{H}or_X,\mathcal{H}}$ can be identified with $\mathfrak{g}/\overline{\mathfrak{s}} \oplus T_{C,x}$, and the cotangent space $T^*_{\mathcal{H}or_X,\mathcal{H}}$ is isomorphic to

$$\overline{\mathfrak{s}}^{\perp} \oplus T^*_{C,x} \cong (\mathfrak{a} + \mathfrak{p}_u \cap \overline{\mathfrak{m}} + \overline{\mathfrak{q}}_u) \oplus T^*_{C,x}.$$

From this description we see that our problem is reduced to the study of $P_u \cap \overline{Q}$-orbits in $\overline{\mathfrak{s}}^{\perp}$. Now Lemma 5.16 completes the proof of the first part of the proposition.

Let us consider the subvariety $\mathcal{H}or^{tr}_X$ of the variety of horospheres $\mathcal{H}or_X$ equal to $(P_u \cap \overline{Q}_u^-) \times Z$. It defines a family of horospheres $\mathcal{U}|_{\mathcal{H}or^{tr}_X} = p^{-1}_{\mathcal{H}or_X}(\mathcal{H}or^{tr}_X)$ that maps isomorphically to X_0 under p_X. Indeed, it consists of translates of the orbits $(P_u \cap \overline{Q})z$ for $z \in Z$ by the elements of $P_u \cap \overline{Q}_u^-$. Moreover from the freeness of the P_u-action and the equality $(P_u \cap \overline{Q}_u^-)(P_u \cap \overline{Q}) = P_u$ (see [7, Prop. 28.7]) it follows that these translates do not intersect pairwise. Hence this family maps isomorphically to the open set $(P_u \cap \overline{Q}_u^-)(P_u \cap \overline{Q})Z = P_u Z = X^\circ$. For the diagonal embedding of Z in $\mathcal{U} \subset G *_{\overline{P}} (X \times Z)$, we get $p^{-1}_{\mathcal{H}or_X}(\mathcal{H}or^{tr}_X) = P_u Z$.

Lemma 6.5. *For $x \in Z$ the subvarieties $\mathcal{H}or^{tr}_X$ and $\mathcal{H}or_{X,x}$ of the variety $\mathcal{H}or_X$ are transversal in the point $\mathcal{H}$ corresponding to the horosphere $\overline{U}x$. We have the following equality for tangent spaces in the point $\mathcal{H} \in \mathcal{H}or_X$:*

$$T_{\mathcal{H}or_X,\mathcal{H}} = T_{\mathcal{H}or_{X,x},\mathcal{H}} \oplus T_{\mathcal{H}or^{tr}_X,\mathcal{H}}. \qquad (*)$$

Proof. Let us notice that the varieties $X, \mathcal{H}or_X, \mathcal{U}$ and the morphisms $p_X, p_{\mathcal{H}or_X}$ are submersive. Let us also notice that the horospheres parameterized by $\mathcal{H}or^{tr}_X$ do not intersect each other and cover the open subset X°. This implies that $p^{-1}_{\mathcal{H}or_X}(\mathcal{H}or^{tr}_X)$ maps isomorphically to X° under p_X. Since p_X is submersive, $p^{-1}_{\mathcal{H}or_X}(\mathcal{H}or^{tr}_X)$ also intersects each fiber $\mathcal{H}or_{X,x} = p_X^{-1}(x)$ for each $x \in X^\circ$ transversally, exactly in one point. Since each fiber $\mathcal{H}or_{X,x}$ maps immersively into $\mathcal{H}or_X$ under $p_{\mathcal{H}or_X}$, this implies the transversality of $\mathcal{H}or^{tr}_X$ and $\mathcal{H}or_{X,x}$ in $\mathcal{H}or_X$. The varieties $p^{-1}_{\mathcal{H}or_X}(\mathcal{H}or^{tr}_X)$ and $\mathcal{H}or_{X,x}$ have complementary dimensions in $\mathcal{U}$, so the varieties $\mathcal{H}or^{tr}_X$ and $\mathcal{H}or_{X,x}$ have complementary dimensions in $\mathcal{H}or_X$, which implies $(*)$. ∎

Lemma 6.6. *$T_{\mathcal{H}or^{tr}_X,\mathcal{H}}$ is identified canonically with $\mathfrak{p}_u \cap \overline{\mathfrak{q}}_u^- \oplus T_{Z,x} \subset \mathfrak{g}/\overline{\mathfrak{s}} \oplus T_{C,x}$ and the fiber of the conormal bundle $N^*_{\mathcal{H}or_X/\mathcal{H}or^{tr}_X,\mathcal{H}}$ is identified with $\mathfrak{p}_u \cap \overline{\mathfrak{q}} \subset \overline{\mathfrak{s}}^\perp \oplus T_{C,x}$.*

Proof. The first assertion is trivial. Since $T_{\mathcal{H}or^{tr}_X,\mathcal{H}} \cong \mathfrak{p}_u \cap \overline{\mathfrak{q}}_u^- \oplus T_{Z,x} \subset \mathfrak{g}/\overline{\mathfrak{s}} \oplus T_{C,x}$, the linear space $N^*_{\mathcal{H}or_X/\mathcal{H}or^{tr}_X,\mathcal{H}}$ is identified with the subspace $\mathfrak{p}_u \cap \overline{\mathfrak{q}} \subset \overline{\mathfrak{s}}^\perp$, which consists of the linear functions on $\mathfrak{g}/\overline{\mathfrak{s}}$ annihilated on $\mathfrak{p}_u \cap \overline{\mathfrak{q}}_u^-$. ■

Dualizing the equality $(*)$ we get:

$$T^*_{\mathcal{H}or_X,\mathcal{H}} = N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}} \oplus N^*_{\mathcal{H}or_X/\mathcal{H}or^{tr}_X,\mathcal{H}} = N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}} \oplus (\mathfrak{p}_u \cap \overline{\mathfrak{q}}). \qquad (**)$$

The proof of Theorem 6.2 will be finished after proving the next proposition.

Proposition 6.7. *The intersection of $N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}}$ and a general $P_u \cap \overline{Q}$-orbit from $\overline{\mathfrak{s}}^\perp \oplus T^*_{C,x}$ consists of a single point.*

Proof.

Let us take a sufficiently general point $\xi \in N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}}$ (we assume that the projection to $\mathfrak{a}$ is sufficiently general). We shall prove that $u\xi \notin N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}}$ for any nontrivial $u \in P_u \cap \overline{Q}$. We represent u via the exponential map $u = \exp(\eta)$, where $\eta = \sum_{\alpha\in\Delta(\mathfrak{p}_u\cap\overline{\mathfrak{q}})} c_\alpha e_\alpha \in \mathfrak{p}_u \cap \overline{\mathfrak{q}}$. Consider the one-parameter subgroup $\lambda\colon \mathbb{K}^\times \to Z(L_0)$ from the proof of Theorem 5.5. We recall that λ is nonnegative on $\overline{\mathfrak{s}}^\perp = \mathfrak{a} + \mathfrak{p}_u \cap \overline{\mathfrak{m}} + \overline{\mathfrak{q}}_u$ and $\mathfrak{a} + \mathfrak{p}_u \cap \overline{\mathfrak{m}}$ is the component of $\overline{\mathfrak{s}}^\perp$ of zero λ-weight. Since λ lies in the stabilizer of $x \in Z$ it preserves the subvariety $\mathcal{H}or_{X,x}$; it also stabilizes the horosphere $\mathcal{H} = (P_u \cap \overline{Q})x$. Consequently, λ acts on the linear space $N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}}$. Let us choose a strictly dominant one-parameter subgroup $\lambda_0 : \mathbb{K}^\times \to T$ (in particular $\langle \lambda_0, \alpha\rangle > 0$ for all $\alpha \in \Delta(\mathfrak{p}_u)$).

Consider the set of $\alpha \in \Delta(\mathfrak{p}_u \cap \overline{\mathfrak{q}})$ such that $c_\alpha \neq 0$ and the value $\ell = \langle \lambda, \alpha\rangle$ is the least possible. We choose some γ from this set with the smallest value $\langle \lambda_0, \gamma\rangle$. For a vector $v \in N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}}$, we denote by v_ℓ the component of weight ℓ with respect to λ. Let us prove the following lemma.

Lemma 6.8. *The T-equivariant projection of the vector $u\xi - \xi$ to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ is zero but $(u\xi - \xi)_\ell \neq 0$.*

Proof. In the proof of this lemma we shall work only with the projection of ξ to $\mathfrak{s}^\perp$ (since the action of $P_u \cap Q$ is trivial on the direct summand $T^*_{C,x}$), so let us assume that $\xi \in \mathfrak{s}^\perp$. First assume that $\ell \neq 0$. Denote by $\xi_{\mathfrak{a}}$ and by $\xi_{\mathfrak{p}_u\cap\overline{\mathfrak{m}}}$ the projection of ξ to $\mathfrak{a}$ and $\mathfrak{p}_u \cap \overline{\mathfrak{m}}$, respectively. The component of ξ of λ-weight zero is equal to $\xi_0 = \xi_{\mathfrak{a}} + \xi_{\mathfrak{p}_u\cap\overline{\mathfrak{m}}}$.

Let us notice that

$$\mathrm{Ad}(u)\xi = \xi + [\eta, \xi] + \frac{1}{2!}[\eta, [\eta, \xi]] + \dots$$

The weights in the λ-weight decomposition of ξ are nonnegative. We recall that ℓ has minimal possible value on the root subspaces in the exponential decomposition of u and application of each e_α increases the λ-weight by $\langle\lambda,\alpha\rangle \geqslant \ell$. Thus $(\mathrm{Ad}(u)\xi)_\ell$ does not contain the summands that consist of monomials in $\mathrm{ad}(e_\alpha)$ applied to the components of ξ with the λ-weight > 0 or of monomials of degree > 1 applied to ξ_0. This implies that $(\mathrm{Ad}(u)\xi)_\ell = \xi_\ell + [\eta_\ell, \xi_0]$. Since $\eta_\ell \in \mathfrak{p}_u$ and $\xi_0 \in \mathfrak{a} + \mathfrak{p}_u$ then $[\eta_\ell, \xi_0] \in \mathfrak{p}_u$. In particular, $[\eta_\ell, \xi_0]$ has zero projection to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$; this proves the first part of the lemma in the case $\ell \neq 0$. From the above we also see that the component of $\mathrm{Ad}(u)\xi - \xi$ of weight γ is equal to $[e_\gamma, \xi_{\mathfrak{a}}]$. Indeed, there are no other components of this T-weight since the application of elements $\mathrm{ad}(e_\alpha)$ (with α such that $\langle\lambda_0,\alpha\rangle \geqslant \langle\lambda_0,\gamma\rangle$) to the components of ξ which belong to $\mathfrak{p}_u \cap \overline{\mathfrak{m}}$ (that have λ_0-weight strictly bigger than zero) gives rise to components with λ_0-weight strictly bigger than $\langle\lambda_0,\gamma\rangle$. The component $[e_\gamma, \xi_{\mathfrak{a}}]$ is nonzero since $(\mathfrak{p}_u \cap \overline{\mathfrak{q}}) \cap \mathfrak{m} = 0$.

If $\ell = 0$ we see that the projection of $(P_u \cap \overline{Q})\xi$ to the eigenspace of zero λ-weight is equal to $(P_u \cap \overline{M})\xi_0 = \xi_0 + \mathfrak{p}_u \cap \overline{\mathfrak{m}}$ and the stabilizer of ξ_0 in $P_u \cap \overline{M}$ is trivial. This implies that $\mathrm{Ad}(u)\xi - \xi$ has nonzero projection to $\mathfrak{p}_u \cap \overline{\mathfrak{m}}$ and its projection to $\mathfrak{a}$ is trivial. ∎

Assume that $\xi, u\xi \in N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}}$. Since $N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}}$ is λ-invariant the components ξ_ℓ and $(u\xi)_\ell$ of weight ℓ with respect to λ also belong to $N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}}$. By the previous lemma ξ_ℓ and $(u\xi)_\ell$ are different vectors with the same projections to the subspace $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$. We note that ξ and $u\xi$ have the same projection to $T^*_{C,x}$, which implies that ξ_ℓ and $(u\xi)_\ell$ have the same projection to the subspace $(\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u) \oplus T^*_{C,x}$. By $(**)$ the linear subspace $N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}}$ projects isomorphically to $(\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u) \oplus T^*_{C,x}$. In particular there are no distinct vectors in $N^*_{\mathcal{H}or_X/\mathcal{H}or_{X,x},\mathcal{H}}$ with the same projection to $(\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u) \oplus T^*_{C,x}$. We come to a contradiction, which proves the proposition. ∎

We have the following consequence of the proof of Proposition 6.7.

Proposition 6.9. *Let $\mathcal{N}^*_{X,z}$ be the fiber of the conormal bundle to the foliation of degenerate horosperes at some point $z \in Z$ and let $V_z \subseteq \mathcal{N}^*_{X,z}$ be a λ-invariant subspace such that $\mu_X(V_z)$ is mapped isomorphically to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ under the T-equivariant projection to this subspace. Then the intersection of $\mu_X(V_z)$ and a general $P_u \cap \overline{Q}$-orbit from $\overline{\mathfrak{s}}^\perp$ consists of a single point.*

Proof. The proof of Proposition 6.7 goes word by word if we use that $\mu_X(V_z)$ is λ-invariant and maps isomorphically to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ under the T-equivariant projection. ∎

This completes the proof of Proposition 6.7. ∎

Now we finished the proof of Theorem 6.2. ∎

7. The Little Weyl group

By Theorem 5.20 we have the generically finite map $G *_{\overline{P}} \mathcal{N}^*_X \to T^*_X$ which is in fact a rational Galois covering as will be shown in Theorem 7.2. The aim of this section is to prove that the Galois group of this covering is equal to the little Weyl group of X introduced by Knop in [9].

Consider the following commutative diagram.

$$\begin{array}{ccccc} G *_{\overline{P}} \mathcal{N}^*_X & \xrightarrow{\mu_{\mathcal{N}^*}} & G *_{\overline{P}} (\mathfrak{a} + \overline{\mathfrak{p}}_u) & \longrightarrow & \mathfrak{a} \\ \downarrow & & \downarrow & & \downarrow \\ T^*_X & \xrightarrow{\mu_X} & G(\mathfrak{a} + \overline{\mathfrak{p}}_u) & \longrightarrow & \mathfrak{t}/W \end{array}$$

Here the upper right horizontal arrow is a rational quotient by the group G. The lower right arrow is the composition of the categorical quotient by G and the Chevalley isomorphism $\mathfrak{g}/\!\!/G = \mathfrak{t}/W$. The central vertical arrow is $g*\xi \mapsto g\xi$. The map $\mu_{\mathcal{N}^*}$ is given by $g*\xi \mapsto g*\mu_X(\xi)$. It is well defined since μ_X is G-equivariant and in particular $\overline{P}$-equivariant.

The little Weyl group can be defined as follows, after Knop [10, 9]. Consider the fiber product $T^*_X \times_{\mathfrak{t}/W} \mathfrak{a}$. In general it is not irreducible. There is a natural embedding of $\mathcal{N}^*_X$ in $T^*_X \times_{\mathfrak{t}/W} \mathfrak{a}$ that is the product of the inclusion in T^*_X and the map which sends $\eta \in \mathcal{N}^*_X$ to the orthogonal projection of $\mu_X(\eta)$ to $\mathfrak{a}$. Let us denote by $\widehat{T}^*_X$ an irreducible component (which is in fact unique, see [13, Lemma 6.7]) of $T^*_X \times_{\mathfrak{t}/W} \mathfrak{a}$ containing the image of $\mathcal{N}^*_X$. Define the action of the Weyl group $N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ on $T^*_X \times_{\mathfrak{t}/W} \mathfrak{a}$ by its action on the right multiple.

Definition 7.1. The maximal subgroup W_X of $N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ that preserves the irreducible component $\widehat{T}^*_X$ is called the little Weyl group of X.

The aim of this section is to prove the following:

Theorem 7.2. *The map $G *_{\overline{P}} \mathcal{N}^*_X \to T^*_X$ is a rational Galois covering with group W_X.*

To prove Theorem 7.2 we need the notion of the normalized moment map $\widetilde{\mu}_X \colon T^*_X \to M_X$ introduced by Knop in [9]. It can be defined via taking the Stein factorization $T^*_X \to M_X \to G(\mathfrak{a}+\mathfrak{p}_u^-)$ of the moment map μ_X. In other words, we take for M_X the normalization of $G(\mathfrak{a}+\mathfrak{p}_u^-)$ in the field of rational functions of T^*_X. We recall that for the horospherical variety G/P_0^- the variety $G*_{P^-}(\mathfrak{a}+\mathfrak{p}_u^-)$ is a G-birational model for M_{G/P_0^-} (see [9, §4]).

The next lemma provides different birational G-models of M_{G/P_0^-}.

Lemma 7.3. *The varieties $G*_{P^-}(\mathfrak{a}+\mathfrak{p}_u^-)$ and $G*_{\overline{P}}(\mathfrak{a}+\overline{\mathfrak{p}}_u)$ are birationally isomorphic to $G*_M(\mathfrak{a}+M*_{M\cap P^-}(\mathfrak{p}_u^-\cap\mathfrak{m}))$ as G-varieties.*

Proof. By Proposition 5.4 there exists $\xi_n \in \mathfrak{p}_u^- \cap \mathfrak{m}$ such that $(P^-\cap M)\xi_n$ is dense in $\mathfrak{p}_u^-\cap\mathfrak{m}$. Since $P_u^-(\mathfrak{a}+(\mathfrak{p}_u^-\cap\mathfrak{m}))$ is dense in $\mathfrak{a}+\mathfrak{p}_u^-$ we get that $P^-(\mathfrak{a}+\xi_n) = P_u^-(P^-\cap M)(\mathfrak{a}+\xi_n)$ is also dense there. This implies that $\mathfrak{a}+\xi_n$ intersects a general P^--orbit from $\mathfrak{a}+\mathfrak{p}_u^-$ in exactly one point. Since $Z_{P^-}(\mathfrak{a}+\xi_n) = Z_{M\cap P^-}(\xi_n)$ we get that the map $P^-*_{Z_{M\cap P^-}(\xi_n)}(\mathfrak{a}+\xi_n) \to \mathfrak{a}+\mathfrak{p}_u$ is birational. Thus $G*_{P^-}(\mathfrak{a}+\mathfrak{p}_u^-)$ is G-birational to $G*_{P^-}(P^-*_{Z_{M\cap P^-}(\xi_n)}(\mathfrak{a}+\xi_n)) \cong G*_{Z_{M\cap P^-}(\xi_n)}(\mathfrak{a}+\xi_n)$

which is contained in $G *_M (\mathfrak{a} + M *_{M\cap P^-} (\mathfrak{p}_u^- \cap \mathfrak{m}))$ as a dense subset.

Using the equality $(\overline{P} \cap M) = (P^- \cap M)$ and repeating literally all the above arguments for the group $\overline{P}$ instead of P^-, we see that $\mathfrak{a} + \xi_n$ also provides a section for the action of $\overline{P}$ on $\mathfrak{a} + \overline{\mathfrak{p}}_u$ and we get a G-equivariant birational isomorphism between $G *_{\overline{P}} (\mathfrak{a} + \overline{\mathfrak{p}}_u)$ and $G *_M (\mathfrak{a} + M *_{M\cap\overline{P}} (\overline{\mathfrak{p}}_u \cap \mathfrak{m}))$. ∎

Proposition 7.4. *The general fibers of the morphisms* $\mu_X : \mathcal{N}_X^* \to \mathfrak{a} + \overline{\mathfrak{p}}_u$ *and* $\mu_{\mathcal{N}^*} : G *_{\overline{P}} \mathcal{N}_X^* \to G *_{\overline{P}} (\mathfrak{a} + \overline{\mathfrak{p}}_u)$ *are irreducible.*

Proof. We shall need the following lemma.

Lemma 7.5. *Let* X *be a normal variety and let* $f : X \to Y$ *be a dominant morphism. Assume we have a rational section of* f*, i.e.* $\sigma : Y \dashrightarrow X$*, such that* $f \circ \sigma = id_Y$*. Then a general fiber of* f *is irreducible.*

Proof. Consider the variety $\widetilde{Y}$ that is equal to the normalization of Y in the field of rational functions on X. Then we have two morphisms $\widetilde{f} : X \to \widetilde{Y}$ and $\pi : \widetilde{Y} \to Y$ such that $f = \pi \circ \widetilde{f}$, a general fiber of $\widetilde{f}$ is irreducible, and π is finite. Then the composition $\widetilde{f} \circ \sigma$ gives a rational section of the finite morphism π, which proves that π is birational and gives the irreducibility of a general fiber of f. ∎

To prove the irreducibility of a general fiber for $\mu_X|_{\mathcal{N}_X^*}$, by Lemma 7.5 it is sufficient to construct a rational section $\mathfrak{a} + \overline{\mathfrak{p}}_u \dashrightarrow \mathcal{N}_X^*$ of the morphism $\mu_X|_{\mathcal{N}_X^*} : \mathcal{N}_X^* \to \mathfrak{a} + \overline{\mathfrak{p}}_u$. Let us notice that by Proposition 5.13 and Remark 5.14 there exists a subspace $V_z \subset \mathcal{N}_{X,z}^*$ such that $\mu_X(V_z)$ is isomorphic to V_z and $\mu_X(V_z)$ projects isomorphically to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ under the T-equivariant projection from $\overline{\mathfrak{s}}^\perp$ to $\mathfrak{a} + \mathfrak{p}_u^- \cap \overline{\mathfrak{q}}_u$ with the fibers parallel to the subspace $\mathfrak{p}_u \cap \overline{\mathfrak{q}}$. By Proposition 6.9 the map $(P_u \cap \overline{Q}) \times \mu_X(V_z) \to \mathfrak{a} + \overline{\mathfrak{p}}_u$ is birational. Since the map $(P_u \cap \overline{Q}) \times V_z \to (P_u \cap \overline{Q})V_z \subset \mathcal{N}_X^*$ is an isomorphism onto its image, from the G-equivariance of μ_X the variety $(P_u \cap \overline{Q})V_z$ defines a rational section of $\mu_X : \mathcal{N}_X^* \to \mathfrak{a} + \overline{\mathfrak{p}}_u$. Since $\mu_{\mathcal{N}^*}$ is G-equivariant its general fibers are also irreducible. ∎

Let us denote by Θ the irreducible component of $\mu_X^{-1}(\mathfrak{a}^{pr} + (\mathfrak{p}_u^- \cap \mathfrak{m})) \cap \mathcal{N}_X^*$ that maps dominantly to $\mathfrak{a} + (\mathfrak{p}_u^- \cap \mathfrak{m})$ (it is unique by Proposition 7.4). Consider $\Sigma = M\Theta$, it is a component of $\mu_X^{-1}(\mathfrak{a}^{pr} + M(\mathfrak{p}_u^- \cap \mathfrak{m}))$ that maps dominantly to $\mathfrak{a} + M(\mathfrak{p}_u^- \cap \mathfrak{m})$ and intersects $\mathcal{N}_X^*$. Since $G(\mathfrak{a}^{pr} + M(\mathfrak{p}_u^- \cap \mathfrak{m}))$ is dense in $\mu_X(T_X^*)$ we get that $G\Sigma$ is dense in T_X^*. Let $\xi \in \mathfrak{a}^{pr} + M(\mathfrak{p}_u^- \cap \mathfrak{m})$. If $\mathrm{Ad}(g)\xi \in (\mathfrak{a}^{pr} + M(\mathfrak{p}_u^- \cap \mathfrak{m}))$ for some $g \in G$, then from the uniqueness of the Jordan decomposition we get that the semisimple parts of ξ and $\mathrm{Ad}(g)\xi$ are conjugate by g and both lie in $\mathfrak{a}^{pr}$. Thus the set $\{g \in G \mid g\Sigma \cap \Sigma \neq \emptyset\}$ is contained in a finite union of the cosets of M in $N_G(\mathfrak{a})$. By [19, Lemma 2] the morphism $G *_M \Sigma \to T_X^*$ is a rational Galois covering with the Galois group N_X/M, where

$$N_X := \{g \in N_G(\mathfrak{a}) \mid g\overline{\Sigma} = \overline{\Sigma}\},$$

and the action of N_X/M is defined by $nM \circ [g * z] = [gn^{-1} * nz]$ for $n \in N_X$.

Theorem 7.6. *The varieties T^*_X and M_X are G-birationally isomorphic to $G *_{N_X} \Sigma$ and $G *_{N_X} (\mathfrak{a}^{pr} + M *_{P^- \cap M} (\mathfrak{p}_u^- \cap \mathfrak{m}))$, respectively. The map*

$$\Phi\colon M *_{P^- \cap M} \Theta \to \Sigma$$

defined as $\Phi(m * \eta) = m\eta$ for $m \in M$ and $\eta \in \Theta$ is a birational isomorphism, and under these birational identifications, the normalized moment map $\widetilde{\mu}_X : T^*_X \to M_X$ is described on some open subset by the formula $\widetilde{\mu}_X([g * \eta]) = [g * \mu_{\mathcal{N}^*}(\Phi^{-1}(\eta))]$.

Proof. To prove that the morphism Φ is birational it is sufficient to show that it is an isomorphism on the open subset $(P_u \cap M) \times \Theta$ in $M *_{P^- \cap M} \Theta$. Let us notice that the projection of $\mathcal{N}^*_X$ to X is equal to $(P_u \cap \overline{Q})Z$ and $\overline{Q} \cap (P_u \cap M) = \{e\}$. Thus we have an isomorphism $(P_u \cap M)(P_u \cap \overline{Q})Z \cong (P_u \cap M) \times (P_u \cap \overline{Q})Z$, which implies $(P_u \cap M)\mathcal{N}^*_X \cong (P_u \cap M) \times \mathcal{N}^*_X$. In particular, $(P_u \cap M)\Theta \cong (P_u \cap M) \times \Theta$.

We have an M-equivariant rational map

$$\mu_{\mathcal{N}^*} \circ \Phi^{-1}\colon \Sigma \dashrightarrow M *_{P^- \cap M} (\mathfrak{a}^{pr} + (\mathfrak{p}_u^- \cap \mathfrak{m})),$$

which induces the rational map

$$\widetilde{\mu}'_X\colon G *_{N_X} \Sigma \dashrightarrow G *_{N_X} (\mathfrak{a}^{pr} + M *_{P^- \cap M} (\mathfrak{p}_u^- \cap \mathfrak{m})),$$

which factors the moment map μ_X. To prove that M_X is birational to

$$G *_{N_X} (\mathfrak{a}^{pr} + M *_{P^- \cap M} (\mathfrak{p}_u^- \cap \mathfrak{m}))$$

and that $\widetilde{\mu}'_X = \widetilde{\mu}_X$ it remains to prove the irreducibility of general fibers for $\widetilde{\mu}'_X$. This follows from the irreducibility of general fibers for $\mu_X|_\Theta : \Theta \to \mathfrak{a}^{\mathrm{pr}} + (\mathfrak{p}_u^- \cap \mathfrak{m})$ (see Proposition 7.4). ■

Remark 7.7. Since $G\mathcal{N}^*_X$ is dense in T^*_X, to study the stabilizer of a general cotangent vector $\eta_x \in T^*_{X,x}$ it suffices to consider $\eta_x \in \mathcal{N}^*_X$, and acting by $\overline{P}$ we may assume that $\xi = \mu_X(\eta_x) \in \mathfrak{a}^{pr} + (\overline{\mathfrak{p}}_u \cap \mathfrak{m})$. From the description of M_X we get that $G_{\eta_x} \subset \overline{S} \cap G_\xi$ and $G^0_{\eta_x} = (\overline{S} \cap G_\xi)^0$.

Proof. [Proof of Theorem 7.2] As we have seen, the Galois group of the rational covering $G *_{\overline{P}} \mathcal{N}^*_X \to T^*_X$ is equal to N_X/M, and by Theorem 7.6 the map $M_{G/P_0^-} \to M_X$ is the quotient by the action of N_X/M, which commutes with the G-action. By the definition of Knop [9] the little Weyl group is the Galois group of the covering $M_{G/P_0^-}/\!\!/G \to M_X/\!\!/G$ (we note that $M_{G/P_0^-}/\!\!/G \cong \mathfrak{a}$) which is also equal to N_X/M. ■

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