Basic Relative Invariants of Homogeneous Cones

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Abstract. The purpose of this paper is to give an explicit expression to the basic relative invariants of a homogeneous cone in terms of the polynomials introduced by Vinberg in 1963. We present a closed formula that exhibits how we obtain the basic relative invariants at one time. We also factorize the determinant of the right multiplication operators of the corresponding clan $V$ by giving an explicit expression to the exponents of the basic relative invariants in terms of the data of $V$.

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Introduction

Homogeneous convex cones in Euclidean vector spaces containing no entire lines (homogeneous cones for short in what follows) correspond in a one-to-one way, up to isomorphisms, to certain non-associative algebras called clans with unit element by Vinberg [8]. Homogeneous cones provide many examples of non-reductive prehomogeneous vector spaces (see [6] for prehomogeneous vector spaces), and as such the associated basic relative invariants are indispensable objects. Here the groups that we consider for the relative invariance are the split solvable Lie groups acting simply transitively on homogeneous cones. In this paper, we study the basic relative invariants of a homogeneous cone $\Omega$ by focusing on an inductive structure of the corresponding clan $V$. During the inductive arguments, there arise naturally representations of clans in the sense of Ishi [4]. The crucial step is to assign an orbit $O$ in the closure $\overline{\Omega}$ of $\Omega$ to every representation $\varphi$. This is done by using some of the results of Graczyk and Ishi [1], and Ishi [2] (see below for details).

Let us describe the contents of this paper in more detail. Let $V$ be a clan of rank $r$ with unit element $e_V$. The product in $V$ is denoted by $\triangle$. Fix an inner product $\langle \cdot | \cdot \rangle$ of $V$ and a complete system $c_1, \ldots, c_r$ of orthogonal primitive idempotents. Then we have the corresponding normal decomposition of $V$ written...
as \( V = \bigoplus_{1 \leq j \leq k \leq r} V_{kj} \). The space \( \mathfrak{h} := \{ L_x; \ x \in V \} \) of left multiplication operators \( L_x \) is a split solvable Lie algebra, and we denote by \( H \) the corresponding connected and simply connected Lie group. The \( H \)-orbit in \( V \) of \( \epsilon \) is a homogeneous cone \( \Omega \) on which \( H \) acts simply transitively. The transpose of the left multiplication operators with respect to \( \langle \cdot, \cdot \rangle \) defines another clan structure \( \triangle \) in \( V \), which we call the dual clan of \( V \). In order to describe the inductive structure of \( V \) mentioned above, we put \( E = \bigoplus_{k>1} V_{k1} \) and \( W = \bigoplus_{1<j\leq k\leq r} V_{kj} \). Then

\[
V = \mathbb{R}c_1 \oplus E \oplus W. \tag{0.1}
\]

General elements \( x \) of \( V \) are denoted by \( \lambda c_1 + \xi + w \) (\( \lambda \in \mathbb{R}, \ \xi \in E, \ w \in W \)) without any comments. We note that \( W \) is a subclan in both structures \((V, \triangle)\) and \((V, \nabla)\). When we consider the clan \((W, \triangle)\), we usually write simply \( W \) for it in the following. We also note \( E \cap W \subset E \), and this enables us to consider the representation \((\varphi, E)\) of \((W, \nabla)\) defined by \( \varphi(w)\xi = \xi \nabla w \) (Proposition 2.1). Let \( Q \) be the bilinear map associated with \( \varphi \) (see (1.10)), and \( \varphi \) the lower triangular part of \( \varphi \) (see (1.8)). Then, by putting \( y = \mu c_1 + \eta + v \), the product \( \triangle \) in \( V \) is described as

\[
x \triangle y = (\lambda \mu)c_1 + (\mu \xi + \frac{1}{2}\lambda \eta + \varphi(x)\eta) + (Q(\xi, \eta) + w \triangle v).
\]

We know by Ishi and Nomura [5] that the irreducible factors of the determinant of the right multiplication operators are the basic relative invariants. This is a key to studying the basic relative invariants by the above inductive structure. We calculate \( \det R_x \) \((x \in V)\) in Proposition 3.2 for the right multiplication operators \( R_x \) of \( V \) to obtain

\[
\det R_x = \lambda^{1 + \dim E - \dim W} \det \left( R_{\lambda w - \frac{1}{2} Q[\xi]}^W \right) \quad (x \in V),
\]

where \( R_{\lambda w}^W \) denotes the right multiplication operator of \( W \) and \( Q[\xi] := Q(\xi, \xi) \). Then we see in Theorem 3.4 that the basic relative invariants \( \Delta^V_j(x), \ldots, \Delta^V_r(x) \) of \( V \) are described by the basic relative invariants \( \Delta^W_j(w), \ldots, \Delta^W_r(w) \) of \( W \) as

\[
\Delta^V_j(x) = \lambda, \quad \Delta^V_j(x) = \lambda^{-\alpha_j} \Delta^W_j(\lambda w - \frac{1}{2} Q[\xi]) \quad (j = 2, \ldots, r), \tag{0.2}
\]

where \( \alpha_2, \ldots, \alpha_r \) are non-negative integers.

To determine the column vector \( \alpha := (\alpha_2, \ldots, \alpha_r) \) with \( \alpha_j \) in (0.2), we need to look at representations of clans more closely. In this paper we do this in a general setting. Let \((\varphi, E)\) be a selfadjoint representation of the dual clan \((V, \nabla)\), anew. By using some of the results on Riesz measures in Graczyk and Ishi [1], and on Gindikin-Riesz distributions in Ishi [2], we can assign, to \( \varphi \), an \( H \)-orbit in the closure \( \overline{\Omega} \) of \( \Omega \). On the other hand, Ishi [2, Theorem 3.5] tells us that the \( H \)-orbits in \( \overline{\Omega} \) are described as \( Hc_\varepsilon \), where \( c_\varepsilon := \varepsilon_1 c_1 + \cdots + \varepsilon_r c_r \) with \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \{0, 1\}^r \). In this way, we attach \( \varepsilon \) to \( \varphi \), and call \( \varphi \) an \( \varepsilon \)-representation.

Applying the above procedure to the representation \((\varphi, E)\) of \((W, \nabla)\) arising from the decomposition (0.1), we take \( \varepsilon = (\varepsilon_2, \ldots, \varepsilon_r) \in \{0, 1\}^{r-1} \) so that \( \varphi \)
is an $\varepsilon$-representation. Then Theorem 5.1 shows that $\alpha$ is described by using the multiplier matrix $\sigma_W$ of $W$ (see (1.4) for definition of the multiplier matrix) as

$$\alpha = \sigma_W(1 - \varepsilon),$$

where $1 = t(1, \ldots, 1)$. Moreover the multiplier matrix $\sigma_V$ of $V$ is written as

$$\sigma_V = \begin{pmatrix} 1 & 0 \\ \sigma_W \varepsilon & \sigma_W \end{pmatrix}.$$  \hfill (0.3)

The formula (0.3) enables us to determine $\sigma_V$ inductively. For any $k = 1, 2, \ldots, r - 1$, let $V^k$ and $E^k$ be the subspaces of $V$ respectively defined by

$$V^k := \bigoplus_{k < l \leq m \leq r} V_{ml}, \quad E^k := \bigoplus_{m > k} V_{mk}. $$

Then $V^k$ are subclans of $(V, \triangledown)$ and $E^k \triangledown V^k \subset E^k$. By the latter property, we have representations $R[k]$ defined by $R[k](x^k) \xi_k := \xi_k \triangledown x^k$ of $V^k$ on $E^k$. We put $\varepsilon^k := \varepsilon(R[k]) \in \{0, 1\}$, and define an $r \times r$ matrix $E_k$ by

$$E_k := \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varepsilon^k & I_{r-k} \end{pmatrix} \quad (k = 1, \ldots, r - 1).$$

Then $\sigma_V$ is described in Theorem 6.1 as

$$\sigma_V = E_{r-1}E_{r-2}\cdots E_1.$$ 

Now that the multiplier matrix $\sigma_V$ is determined, we are able to have an explicit expression of the basic relative invariants $\Delta_V^j(x)$. Introducing the polynomials $D_1(x), \ldots, D_r(x)$ defined by Vinberg [8, the formula (25), p. 385] (see also Ishi [3]), we obtain in Theorem 6.2

$$\Delta_1^V(x) = D_1(x), \quad \Delta_j^V(x) = \frac{D_j(x)}{\prod_{i<j} D_i(x)^{-\sigma_{ji} + \sigma_{j,i+1} + \cdots + \sigma_{jj}}} \quad (j = 2, \ldots, r).$$

Our final objective is to give an expression to the positive integers $n_1, \ldots, n_r$ appearing in the formula

$$\text{Det} R_x = \Delta_1(x)^{n_1} \cdots \Delta_r(x)^{n_r} \quad (x \in V).$$

The row vector $\underline{n} := (n_1, \ldots, n_r)$, called the basic index of $V$ in this paper, is expressed as (Theorem 6.5)

$$\underline{n} = m\sigma_V^{-1},$$

where $m = (m_1, \ldots, m_r)$ is a row vector the entries of which are positive integers coming from the normal decomposition of $V$ (see (6.6) for definition).

We organize this paper as follows. Section 1 collects definitions and facts about clans and homogeneous cones. In Section 2, we study an inductive structure of a clan $V$. Section 3 is devoted to expressing the basic relative invariants of $V$ with non-negative integers $\alpha_j$ as in (0.2). In Section 4, we introduce $\varepsilon$-representations. The non-negative integers $\alpha_j$ in (0.2) are determined in Section 5. In the last section, Section 6, we present the multiplier matrix $\sigma_V$ explicitly, and consequently we obtain an explicit expression of the basic relative invariants of $V$ by the polynomials introduced by Vinberg. We also give an explicit expression to the basic index.
1. Preliminaries

Let $V$ be a finite-dimensional real vector space with a bilinear product $\triangle$. For $x \in V$, we denote by $L_x$ the left multiplication operator $L_xy = x \triangle y$ ($y \in V$). The pair $(V, \triangle)$ (or simply $V$) is called a clan if the following three conditions are satisfied:

(C1) $(V, \triangle)$ is left-symmetric: $L_xL_y - L_yL_x = L_x\triangle y - y \triangle x$ for all $x, y \in V$.
(C2) there exists $s \in V^*$ such that $s(x \triangle y)$ defines an inner product in $V$,
(C3) for each $x \in V$, the operator $L_x$ has only real eigenvalues.

Linear forms $s$ with the property (C2) are said to be admissible. We denote by $R_x$ the right multiplication operator $R_xy = y \triangle x$ ($y \in V$). In this paper, we always assume that a clan has a unit element.

Let $V$ be a clan with unit element $e_V$. The rank of $V$ is denoted by $r$. This means that there is a complete system of orthogonal primitive idempotents $c_1, \ldots, c_r$ with $c_1 + \cdots + c_r = e_V$ such that we have the following decomposition of $V$:

$$V = \bigoplus_{1 \leq j \leq k \leq r} V_{kj}, \quad (1.1)$$

where $V_{jj} = \mathbb{R}c_j$ ($j = 1, \ldots, r$) and

$$V_{kj} := \{ x \in V; \ L_{c_j}x = \frac{1}{2}(\delta_{kj} + \delta_{jk})x, \ R_{c_j}x = \delta_{ij}x \ (i = 1, \ldots, r) \} \quad (j < k).$$

The decomposition (1.1) is called the normal decomposition of $V$ associated with $c_1, \ldots, c_r$. The multiplication rules are

$$V_{ji} \triangle V_{ik} = \{0\} \quad (\text{if } i \neq k, l), \quad V_{kj} \triangle V_{ji} \subset V_{ki},$$

$$V_{ji} \triangle V_{ki} \subset V_{jk} \text{ or } V_{kj} \quad (\text{according to } j \geq k \text{ or } j \leq k). \quad (1.2)$$

By (C1) and (C3), the space $\mathfrak{h} := \{L_x; \ x \in V\}$ of left multiplication operators forms a split solvable Lie algebra. We note here that $\mathfrak{h}$ is linearly isomorphic to $V$. Let $H := \exp \mathfrak{h}$ be the connected and simply connected Lie group corresponding to $\mathfrak{h}$. We denote by $\Omega$ the $H$-orbit in $V$ through $e_V$. We know that $\Omega$ is a proper open convex cone in $V$, and $H$ acts on $\Omega$ simply transitively. For every $\varepsilon := \varepsilon(\varepsilon_1, \ldots, \varepsilon_r) \in \{0, 1\}^r$, we put $c_\varepsilon := \varepsilon_1c_1 + \cdots + \varepsilon_rc_r$. Then we have $c_\varepsilon \in \overline{\Omega}$, the closure of $\Omega$, and let $O_\varepsilon := Hc_\varepsilon$ be the $H$-orbit of $c_\varepsilon$. Note that $O_1 = \Omega$, where $1 = \varepsilon(1, \ldots, 1)$. By Ishi [2, Theorem 3.5], the $H$-orbit decomposition of $\overline{\Omega}$ is described as

$$\overline{\Omega} = \bigsqcup_{\varepsilon \in \{0, 1\}^r} O_\varepsilon.$$

By introducing the lexicographic order among the subspaces $V_{kj}$ in (1.1), we see that every $L_x$ ($x \in V$) is simultaneously represented by a lower triangular matrix. Then for each $h \in H$, there exist unique $h_{jj} > 0$ ($j = 1, \ldots, r$) and $v_{kj} \in V_{kj}$ ($1 \leq j < k \leq r$) such that by setting $T_{jj} := (2 \log h_{jj})L_{c_j}$ and $L_j := \sum_{k>j} L_{v_{kj}}$, we have

$$h = (\exp T_{11})(\exp L_1)(\exp T_{22}) \cdots (\exp L_{r-1})(\exp T_{rr}). \quad (1.3)$$
Theorem 1.1 (Ishi [3]). There exist irreducible relatively $H$-invariant polynomial functions $\Delta_1, \ldots, \Delta_r$ by which any relatively $H$-invariant polynomial function $p$ on $V$ is written as

$$p(x) = (\text{const}) \cdot \Delta_1(x)^{n_1} \cdots \Delta_r(x)^{n_r} \quad ((n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 0}^r).$$

Moreover $\Omega$ is described as

$$\Omega = \{ x \in V; \Delta_1(x) > 0, \ldots, \Delta_r(x) > 0 \}.$$

The polynomials $\Delta_1(x), \ldots, \Delta_r(x)$ are called the basic relative invariants of the cone $\Omega$. They are also called the basic relative invariants of the clan $V$ in this paper. We assume that the numbering of the basic relative invariants is given by the procedure of Ishi [3] according to $e_1, \ldots, e_r$. For $j = 1, \ldots, r$, let $\sigma_j = (\sigma_{j1}, \ldots, \sigma_{jr})$ be the multiplier of the basic relative invariant $\Delta_j(x)$, and we place them in an $r \times r$ matrix as

$$\sigma_V := \begin{pmatrix} \sigma_{1j} \\ \vdots \\ \sigma_{rj} \end{pmatrix} = (\sigma_{jk}). \quad (1.4)$$

In this paper, we call $\sigma_V$ the multiplier matrix of the clan $V$. We note that by the procedure of Ishi [3], the matrix $\sigma_V$ is lower triangular with all $\sigma_{jk} \in \mathbb{Z}_{\geq 0}$ and $\sigma_{jj} = 1$ ($j = 1, \ldots, r$). In particular, $\sigma_V$ is invertible. We also note that the basic relative invariants are homogeneous polynomials, and we have by definition

$$\deg \Delta_j = \sigma_{j1} + \cdots + \sigma_{jj} \quad (j = 1, \ldots, r). \quad (1.5)$$

Now we assume that the inner product $\langle \cdot | \cdot \rangle$ of $V$ is given by an admissible linear form $s_0$. Let us define a bilinear product $\nabla$ in $V$ through

$$\langle x \nabla y | z \rangle = \langle y | x \nabla z \rangle \quad (x, y, z \in V). \quad (1.6)$$

Then it turns out that the product $\nabla$ defines a clan structure in $V$. The clan $(V, \nabla)$ is called the dual clan of $(V, \Delta)$. The linear form $s_0$ is also an admissible linear form for $(V, \nabla)$. In fact, we have $s_0(x \nabla y) = \langle x \nabla y | e_V \rangle = \langle y | x \rangle$. Moreover it is easy to see from (1.6) that $e_V$ is also a unit element of $(V, \nabla)$. The cone corresponding to $(V, \nabla)$ is the dual cone $\Omega^*$ of $\Omega$ with respect to the inner product $\langle \cdot | \cdot \rangle$, where

$$\Omega^* := \{ x \in V; \langle x | y \rangle > 0 \text{ for all } y \in \Omega \setminus \{0\} \}.$$
Proposition 1.2. The following relationships hold between $\triangle$ and $\triangledown$.

(1) For $x, y \in V$, we have $x \triangle y + x \triangledown y = y \triangle x + y \triangledown x$.

(2) For $i = 1, \ldots, r$, one has $L^{-}_i = L^+_i$.

Proof. (1) For any $z \in V$, we have by (C1)

$$\langle x \triangle y - y \triangle x | z \rangle = s_0(\langle x \triangle y - y \triangle x | z \rangle) = s_0(\langle x \triangle (y \triangle z) - y \triangle (x \triangle z) \rangle) = \langle x y \triangledown z - y x \triangledown z \rangle = \langle y \triangledown x - x \triangledown y | z \rangle.$$ 

Hence, the assertion is proved.

(2) Suppose $x_{kj} \in V_{kj}$ $(j \leq k)$. Then for any $y \in V$, we have

$$\langle c_i \triangledown x_{kj} | y \rangle = \langle x_{kj} | c_i \triangle y \rangle = \langle x_{kj} | c_k \triangle y_{kj} \rangle = \frac{1}{2} (\delta_{ij} + \delta_{ik}) \langle x_{kj} | y \rangle,$$

where $y_{kj}$ is the $V_{kj}$-component of $y$. Thus we get $L^{-}_i x_{kj} = \frac{1}{2} (\delta_{ij} + \delta_{ik}) x_{kj} = L^+_i x_{kj}$ for any $x_{kj} \in V_{kj}$. This shows $L^{-}_i = L^+_i$. ■

Proposition 1.2 (2) shows that $c_1, \ldots, c_r$ form also a complete system of orthogonal primitive idempotents of the dual clan $(V, \triangledown)$. We denote by $R^+_x$ the right multiplication operator of $(V, \triangledown)$ by $x \in V$. By (1) and (2) of Proposition 1.2, we get $R^+_i x_{kj} = \delta_{ik} x_{kj}$ for any $x_{kj} \in V_{kj}$ $(j \leq k)$ and $i = 1, \ldots, r$. Thus we have

$$V_{kj} = \{ x \in V; L^+_i x = \frac{1}{2} (\delta_{ij} + \delta_{ik}) x, R^+_i x = \delta_{ik} x (i = 1, \ldots, r) \}.$$ 

This implies that the decomposition (1.1) also serves as a normal decomposition of $(V, \triangledown)$ relative to $c_1, \ldots, c_r$ with the multiplication rules

$$V_{jl} \triangledown V_{ik} = \{ 0 \} \quad \text{(if } j \neq k, l), \quad V_{jl} \triangledown V_{kj} \subset V_{ki}, \quad V_{ki} \triangledown V_{kj} \subset V_{jl} \text{ or } V_{ij} \quad \text{(according to } i \leq j \text{ or } i \geq j). \quad (1.7)$$

For later sections, we give here the definition of a representation of the clan $(V, \triangledown)$. Let $E$ be a real Euclidean vector space with inner product $\langle \cdot | \cdot \rangle_E$. We denote by $\mathcal{L}(E)$ the vector space of all linear operators on $E$. For a linear map $\varphi : V \to \mathcal{L}(E)$, let $\varphi$ and $\varphi$ be the “lower triangular part” and the “upper triangular part” of $\varphi$ associated with $c_1, \ldots, c_r$ respectively given by

$$\varphi(x) := \frac{1}{2} \sum_{j=1}^r x_{jj} \varphi(c_j) + \sum_{j<k} \varphi(c_k) \varphi(x_{kj}) \varphi(c_j),$$

$$\varphi(x) := \frac{1}{2} \sum_{j=1}^r x_{jj} \varphi(c_j) + \sum_{j<k} \varphi(c_j) \varphi(x_{kj}) \varphi(c_k),$$

where we write $x \in V$ as $x = \sum x_{jj} c_j + \sum_{j<k} x_{kj}$ according to (1.1). A linear map $\varphi : V \to \mathcal{L}(E)$ is called a selfadjoint representation of the clan $(V, \triangledown)$ if $\varphi(x)$ is a selfadjoint operator for every $x \in V$ and if the following condition is satisfied:

$$\varphi(x \triangledown y) = \varphi(x) \varphi(y) + \varphi(y) \varphi(x) \quad (x, y \in V). \quad (1.9)$$
We always require that \(\varphi(e_V)\) is the identity operator. In this paper, we only consider selfadjoint representations, and often drop the adjective selfadjoint for simplicity. Associated with \(\varphi\), we define a symmetric bilinear map \(Q: E \times E \to V\) through
\[
\langle \varphi(x)\xi | \eta \rangle_E = \langle Q(\xi, \eta) | x \rangle \quad (\xi, \eta \in E, \ x \in V).
\]
From now on, we put \(Q[\xi] := Q(\xi, \xi)\) and \(Q[E] := \{Q[\xi]; \xi \in E\}\).

2. Inductive structure of a clan

Vinberg [8, Chapter II, Section 4] tells us that any clan has an inductive structure. In this section, we present the inductive structure explicitly by using a representation. Let \(V\) be a clan of rank \(r\) with unit element \(e_V\). We keep to the notation used in Section 1. By the normal decomposition (1.1), we put
\[
E = \bigoplus_{k>1} V_{k1}, \quad W = \bigoplus_{1<j\leq k\leq r} V_{kj}.
\]
Then, we have
\[
V = \mathbb{R}c_1 \oplus E \oplus W.
\]
We denote general elements \(x\) of \(V\) by
\[
\lambda c_1 + \xi + w \quad (\lambda \in \mathbb{R}, \ \xi \in E, \ w \in W)
\]
without any comments. Inner products of \(E\) and \(W\) are taken respectively as the restrictions of the inner product \(\langle \cdot | \cdot \rangle\) of \(V\). The multiplication rules (1.2) and (1.7) tell us that \(W\) is a subclan in both structures \((V, \triangle)\) and \((V, \nabla)\) with unit element \(e_W := e_V - c_1\). When we consider the clan \((W, \triangle)\), we usually write simply \(W\) for it in what follows. We have \(E \nabla W \subset E\) again by (1.7), and this enables us to consider the linear map \(\varphi: W \to \mathcal{L}(E)\) defined by
\[
\varphi(w)\xi = \xi \nabla w \quad (w \in W, \ \xi \in E).
\]

Proposition 2.1. The pair \((\varphi, E)\) is a selfadjoint representation of \((W, \nabla)\).

Proof. Let \(\xi \in E\) and \(w \in W\). The multiplication rules (1.2) yield \(\xi \triangle w = 0\). Thus we have by Proposition 1.2 (1)
\[
\varphi(w)\xi = \xi \nabla w = w \triangle \xi + w \nabla \xi = (L_w + L_w^\nabla)\xi.
\]
Since \(L_w^\nabla = (L_w)^*\) by (1.6), we see that \(\varphi(w)\) is selfadjoint. Furthermore for all \(w, v \in W\) and \(\xi \in E\), we have by (C1) and (2.4)
\[
\varphi(w \nabla v)\xi = w \nabla (\xi \nabla v) + (\xi \nabla w - w \nabla \xi) \nabla v
\]
\[
= w \nabla (\xi \nabla v) + (w \triangle \xi) \nabla v.
\]
Taking the lower and the upper triangular part of \(\varphi(x)\) in (2.4), we see that \(\varphi(w) = L_w\) and \(\varphi(w) = L_w^\nabla\). Then the last term of (2.5) is equal to
\[
(\varphi(w)\varphi(v) + \varphi(v)\varphi(w))\xi.
\]
Since \(\varphi(e_W)\) is obviously the identity operator, the pair \((\varphi, E)\) is now a selfadjoint representation of the clan \((W, \nabla)\).
Let $Q$ be the symmetric bilinear map associated with $\varphi$.

**Proposition 2.2.** The product $\triangle$ in $V$ is described as

$$x \triangle y = (\lambda \mu)c_1 + (\mu \xi + \frac{1}{2} \lambda \eta + \varphi(x)\eta) + (Q(\xi, \eta) + w \triangle v),$$

where we set $y = \mu c_1 + \eta + v \in V$ as we do for $x$ in (2.2).

**Proof.** By definition of $E$, we have $c_1 \triangle \eta = \frac{1}{2} \eta$ and $\xi \triangle c_1 = \xi$. Next, the multiplication rules (1.2) imply that

$$c_1 \triangle W = W \triangle c_1 = E \triangle W = \{0\}.$$ 

By the proof of Proposition 2.1, we obtain $w \triangle \eta = \varphi(w)\eta$. Moreover, we have for all $w \in W$

$$\langle Q(\xi, \eta) | w \rangle = \langle \varphi(w)\xi | \eta \rangle = \langle \xi \triangledown w | \eta \rangle = \langle w | \xi \triangle \eta \rangle.$$ 

Thus $Q$ is described as

$$Q(\xi, \eta) = \xi \triangle \eta \quad (\xi, \eta \in E). \quad (2.6)$$

Since the product $\triangle$ is bilinear, the proof is now completed. ■

We now describe the action of $H$ on $V$ with respect to the decomposition (2.1). Let $h_W := \{L_w; w \in W\}$ be the Lie algebra of left multiplication operators of $W$ and $H_W := \exp h_W$ the corresponding Lie subgroup of $H$. Let $h \in H$ and we write $h$ as in (1.3). Then putting

$$\xi_h = v_{21} + \cdots + v_{r1} \in E, \quad h_W = \exp T_{22} \exp L_2 \cdots \exp T_{rr} \in H_W,$$

we have $h = (\exp T_{11})(\exp L_{\xi_h})h_W$, where $T_{11} = (2 \log h_{11})L_{c_1}$ ($h_{11} > 0$).

**Lemma 2.3.** Let $y = \mu c_1 + \eta + v \in V$. Then

$$h y = \mu(h_{11})^2c_1 + h_{11}(\mu \xi_h + h_W \eta) + \left(\frac{1}{2} \mu Q[\xi_h] + Q(\xi_h, h_W \eta) + h_W v\right).$$

**Proof.** We first note that the multiplication rules (1.2) tell us that

$$h_W c_1 = c_1, \quad h_W \eta \in E, \quad h_W v \in W.$$

Next we have again by (1.2)

$$R_{c_1 | W} = \text{id}_W, \quad E \triangle E \subset W, \quad E \triangle W = \{0\},$$

so that recalling (2.6), we obtain

$$(\exp L_{\xi_h}) c_1 = c_1 + \xi_h \triangle c_1 + \frac{1}{2} \xi_h \triangle (\xi_h \triangle c_1) = c_1 + \xi_h + \frac{1}{2} Q[\xi_h],$$

$$(\exp L_{\xi_h}) \eta = \eta + Q(\xi_h, \eta), \quad (\exp L_{\xi_h}) v = v.$$
Finally,
\[(\exp T_{11})c_1 = (h_{11})^2c_1, \quad (\exp T_{11})\eta = h_{11}\eta, \quad (\exp T_{11})v = v.\]

These observations yield that
\[hc_1 = \exp T_{11}(c_1 + \xi_h + \frac{1}{2}Q[\xi_h]) = (h_{11})^2c_1 + h_{11}\xi_h + \frac{1}{2}Q[\xi_h],\]
\[h\eta = \exp T_{11}(h_{11}\eta + Q(\xi_h, h_{11}\eta)) = h_{11}h_{11}\eta + Q(\xi_h, h_{11}\eta),\]
\[hv = h_{11}v.\]

Hence we arrive at the formula in the lemma. \[\blacksquare\]

3. Inductive description of basic relative invariants

In this section, we study the basic relative invariants by considering the right multiplication operators of clans. The reason for this is the following theorem.

**Theorem 3.1** (Ishi–Nomura [5]). *The basic relative invariants are the irreducible factors of the determinant of the right multiplication operators.*

Let \(V\) be a clan of rank \(r\) and we decompose \(V\) as in (2.1). We keep to the notation of the previous sections. By Proposition 2.2, the right multiplication operators \(R_x\) by \(x \in V\) of the clan \(V\) are described as
\[R_x = \begin{pmatrix}
\lambda & 0 & 0 \\
\frac{1}{2} \xi & \lambda \id_E & R_\xi \\
0 & R_\xi & R^W_w
\end{pmatrix},\]
where \(R^W_w\) is the right multiplication operator of \(W\), and we note that \(R_\xi(W) \subset E\) and \(R_\xi(E) \subset W\). As in [7, Proposition 4.1], we have the following proposition.

**Proposition 3.2.** *For \(x \in V\), one has*
\[\Det R_x = \lambda^{1+\dim E-\dim W} \Det(R^W_w).\]

Let \(\Delta^W_2(w), \ldots, \Delta^W_r(w)\) be the basic relative invariants of \(W\), where the labeling \(2, \ldots, r\) is in accordance with the complete system of orthogonal primitive idempotents \(c_2, \ldots, c_r\) of \(W\). Noting that the basic relative invariants are homogeneous polynomials, we denote by \(d_j\) the homogeneous degree of \(\Delta^W_j(w)\).

**Theorem 3.1** and **Proposition 3.2** tell us that the basic relative invariants of \(V\) are exhausted by the polynomial \(\lambda\) and the irreducible factors of \(\Delta^W_j(\lambda w - \frac{1}{2}Q[\xi])\) \((j = 2, \ldots, r)\).

**Proposition 3.3.** *For each \(j = 2, \ldots, r\), the only possible factor of the polynomial \(\Delta^W_j(\lambda w - \frac{1}{2}Q[\xi])\) is \(\lambda^{\alpha_j}\) for some non-negative integer \(\alpha_j \leq d_j\).*

**Proof.** We put
\[P_j(x) := \Delta^W_j(\lambda w - \frac{1}{2}Q[\xi]) \quad (j = 2, \ldots, r).\]
Since \( x = \lambda c_1 + \xi + w \mapsto \lambda w - \frac{1}{2}Q[\xi] \) is a quadratic map, the polynomial \( P_j(x) \) is a homogeneous polynomial of degree 2\( d_j \). Moreover, in each monomial of \( P_j(x) \), the degrees of \( \lambda \) and \( w \) are the same. In particular, since \( \Delta^W_j(\lambda w) = \lambda^{d_j} \Delta^W_j(w) \), the degree of the polynomial \( P_j(x) \) with respect to \( \lambda \) is \( d_j \), and the coefficient of \( \lambda^{d_j} \) is the irreducible polynomial \( \Delta^W_j(w) \). Hence \( P_j(x) \) is written as

\[
P_j(x) = \lambda^{d_j} \Delta^W_j(w) + \lambda^{d_j-1} p_j^{(1)}(w, \xi) + \cdots + p_j^{(d_j)}(w, \xi),
\]

where each \( p_j^{(k)}(w, \xi) \) is a polynomial of \( w \) and \( \xi \) of degree \( d_j + k \), and its degree with respect to \( w \) is strictly lower than \( d_j \). Since \( \Delta^W_j(w) \) is irreducible, \( P_j(x) \) is factorized as

\[
P_j(x) = \lambda^{\alpha_j} (\lambda^{d_j-\alpha_j} \Delta^W_j(w) + \lambda^{d_j-\alpha_j-1} p_j^{(1)}(w, \xi) + \cdots + p_j^{(d_j-\alpha_j)}(w, \xi)), \tag{3.1}
\]

where \( \alpha_j \) is the maximal non-negative integer such that \( p_j^{(d_j-\alpha_j)} \neq 0 \). Thus \( \lambda^{-\alpha_j} P_j(x) \) is irreducible. The proof is now completed.

Propositions 3.2 and 3.3 immediately give the following theorem.

**Theorem 3.4.** The basic relative invariants \( \Delta^V_j(x) \) \( (j = 1, \ldots, r) \) of \( V \) are given by

\[
\Delta^V_j(x) = \begin{cases} 
\lambda & (j = 1), \\
\lambda^{-\alpha_j} \Delta^W_j(\lambda w - \frac{1}{2}Q[\xi]) & (j = 2, \ldots, r).
\end{cases} \tag{3.2}
\]

### 4. \( \varepsilon \)-representations

In order to determine the non-negative integers \( \alpha_j \) in Theorem 3.4, we need to investigate the relationship between representations of clans and the range of the corresponding quadratic maps. Let \( V \) be a clan of rank \( r \) and \( (\varphi, E) \) a representation of \( (V, \nabla) \). We maintain the notation in Section 1. We begin with the following lemma.

**Lemma 4.1.** For \( x \in V \) and \( \xi, \eta \in E \), one has

\[
(\exp L_x)Q(\xi, \eta) = Q((\exp \varphi(x))\xi, (\exp \varphi(x))\eta).
\]

**Proof.** Recalling the definition of \( \varphi(x) \) in (1.8), we have \( \varphi(x)^* = \varphi(x) \). Thus for any \( y \in V \) we have by (1.9)

\[
\langle L_x Q(\xi, \eta) | y \rangle = \langle Q(\xi, \eta) | x \nabla y \rangle = \langle \varphi(x \nabla y)\xi | \eta \rangle_E
\]

\[
= \langle \varphi(x)\varphi(y)\xi + \varphi(y)\varphi(x)\xi | \eta \rangle_E
\]

\[
= \langle \varphi(y)\xi | \varphi(x)\eta \rangle_E + \langle \varphi(y)\varphi(x)\xi | \eta \rangle_E
\]

\[
= \langle Q(\xi, \varphi(x)\eta) + Q(\varphi(x)\xi, \eta) | y \rangle.
\]

This equation yields

\[
L_x Q(\xi, \eta) = Q(\varphi(x)\xi, \eta) + Q(\xi, \varphi(x)\eta) \quad (x \in V, \xi, \eta \in E).
\]

Thus we obtain the lemma.
Corollary 4.2. \( Q \) is \( \Omega \)-positive, that is, \( Q[\xi] \in \overline{\Omega}\{0\} \) for all \( \xi \in E\{0\} \).

**Proof.** For each \( x \in \Omega^* \), we take \( x_0 \in V \) such that \( (\exp L_{x_0}^\gamma)e_V = x \). Since \( L_{x_0}^\gamma = (L_{x_0})^* \), we have by Lemma 4.1 and (1.10)
\[
\langle Q[\xi] | x \rangle = \langle (\exp L_{x_0})Q[\xi] | e_V \rangle = \langle Q[(\exp \varphi(x_0))\xi] | e_V \rangle = \|((\exp \varphi(x_0))\xi)\|_E^2 \geq 0,
\]
where \( \|\xi\|_E^2 := \langle \xi | \xi \rangle_E \) is the norm of \( E \). Hence \( Q[\xi] \in \overline{\Omega} \). Moreover, since \( \exp \varphi(x_0) \) is invertible, we see that \( Q[\xi] = 0 \) if and only if \( \xi = 0 \). Thus the corollary is proved.

Let \( (\varphi, E) \) be any representation of \( (V, \nabla) \) and \( Q \) the corresponding bilinear map. The Riesz measure \( \mu_Q \) associated with the quadratic map \( Q[\xi] \) is, by definition, the image of the Lebesgue measure \( d\xi \) on \( E \) by \( Q[\xi] \) (cf. Graczyk and Ishi [1]). In other words, we have for any measurable function \( f \) on \( V \)
\[
\int_V f(x)\mu_Q(dx) = \int_E f(Q[\xi])d\xi.
\]
Since \( Q \) is \( \Omega \)-positive by Corollary 4.2, the Riesz measure \( \mu_Q \) is supported by \( \overline{\Omega} \). We now show that to \( \varphi \) we can assign an \( H \)-orbit \( O_\varepsilon \) in \( \overline{\Omega} \) (\( \varepsilon \in \{0,1\}^r \)). In this case, we say that \( \varphi \) is an \( \varepsilon \)-representation.

Now Lemma 4.1 and (1.10) yield that for any \( x \in V \) and \( h \in H \)
\[
\text{Det} \varphi(h^*x) = \text{Det}(h^*\varphi(x)h) = (\text{Det} h)^2 \text{Det} \varphi(x).
\]
Thus \( \text{Det} \varphi(x) \) is a relatively \( H \)-invariant polynomial and its multiplier \( l = \{l_1, \ldots, l_r\} \in \mathbb{Z}_{\geq 0}^r \) satisfies
\[
\text{Det} \varphi(\lambda_1 c_1 + \cdots + \lambda_r c_r) = (\lambda_1)^{l_1} \cdots (\lambda_r)^{l_r} \quad (\lambda_1, \ldots, \lambda_r \in \mathbb{R})
\]
with \( l_j = \dim \varphi(c_j)E \) (\( j = 1, \ldots, r \)). Let \( \mathcal{R}_s \) (\( s \in \mathbb{R}^r \)) be the Gindikin–Riesz distribution defined in Ishi [2] (in that paper, it is simply called the Riesz distribution). Then, by [1, (3.29)], we have
\[
\mu_Q = a^{\dim E/2} \mathcal{R}_{l/2}.
\]
(4.1)
Since \( \mu_Q \) is a positive measure, so is \( \mathcal{R}_{l/2} \). Let \( \Xi \) be the Gindikin–Wallach set, which is the set of \( s \) for \( \mathcal{R}_s \) to be a positive measure (see Ishi [2, Theorem 6.2]). Thus by (4.1), we obtain \( l/2 \in \Xi \). Putting \( d_{kj} := \dim V_{kj} \) for \( 1 \leq j < k \leq r \), we define \( l(i) \in \mathbb{R}^r \) (\( j = 1, \ldots, r \)) inductively by \( l(1) := l \) and, for \( i = 1, \ldots, r-1 \),
\[
l(i+1) := \begin{cases} 
  l(i) - l(0, \ldots, 0, d_{i+1,i}, \ldots, d_{ri}) & \text{if } l(i) > 0, \\
  l(i) & \text{if } l(i) \leq 0.
\end{cases}
\]
Further we define \( \varepsilon(\varphi) = \{\varepsilon_1, \ldots, \varepsilon_r\} \in \{0,1\}^r \) by
\[
\varepsilon_i = \begin{cases} 
  1 & \text{if } l(i) > 0, \\
  0 & \text{if } l(i) \leq 0
\end{cases} \quad (i = 1, \ldots, r).
\]
Then by [2, p. 183], \( \mathcal{R}_{l/2} \) is a measure on \( O_{\varepsilon(\varphi)} \). Using Proposition 3.10 and Theorem 3.13 of [1], we see that the support of \( \mathcal{R}_{l/2} \) is equal to \( \overline{O}_{\varepsilon(\varphi)} \), the closure of \( O_{\varepsilon(\varphi)} \). These observations together with (4.1) give the following proposition.

**Proposition 4.3.** \( \varphi \) is an \( \varepsilon(\varphi) \)-representation.
5. Calculation of $\alpha_j$

In this section, we determine the non-negative integers $\alpha_2, \ldots, \alpha_r$ that appeared in Theorem 3.4. To do so, we return to the representation $(\varphi, E)$ of $(W, \triangledown)$ defined by (2.3). In particular, the orbits which we consider are the $H_W$-orbits in $W$. Let $e := e_W$ be the unit element of $W$. Let $\Omega_W$ be the $H_W$-orbit of $e$ in $W$. Then $\Omega_W$ is the homogeneous cone corresponding to the clan $(W, \triangle)$. Let us put $e = e(\varphi) = t(\varepsilon_2, \ldots, \varepsilon_r) \in \{0, 1\}^{r-1}$ so that $\varphi$ is an $e$-representation. The corresponding quadratic map $Q[\xi]$ satisfies $Q[E] = \overline{O}_e$, where $O_e$ is the $H_W$-orbit through $c_e := \varepsilon_2 c_2 + \cdots + \varepsilon_r c_r$ in the closure $\overline{\Omega}_W$ of $\Omega_W$.

Let us consider the polynomials $\Delta_j^W(\lambda w - w_e)$ $(\lambda \in \mathbb{R}, w \in W, w_e \in \overline{O}_e)$. If $w \in \Omega_W$, then putting $w = h e$ with $h \in H_W$, we have by the relative invariance

$$\Delta_j^W(\lambda w - w_e) = \Delta_j^W(w) \Delta_j^W(\lambda e - h^{-1}w_e). \tag{5.1}$$

For $j = 2, \ldots, r$, we put

$$\overline{P}_j^e(\lambda, w_e) := \Delta_j^W(\lambda e - w_e) = \lambda^{d_j} + \lambda^{d_j-1} q_j^{(1)}(w_e) + \cdots + q_j^{(d_j)}(w_e), \tag{5.2}$$

where $q_j^{(k)} (k = 1, \ldots, d_j)$ is a polynomial function on $\overline{O}_e$ of degree $k$. By the coefficient comparison of (5.1) with (3.1) relative to $\lambda$, the polynomials $q_j^{(k)}(w_e)$ are the zero-polynomials on $\overline{O}_e$ for $k = d_j - \alpha_j + 1, \ldots, d_j$, and the polynomial $q_j^{(d_j-\alpha_j)}(w_e)$ is non-zero. In particular, $\lambda^{-\alpha_j} \overline{P}_j^e(\lambda, w_e)$ is an irreducible polynomial. By (5.1), we see that $\lambda^{-\alpha_j} \Delta_j^W(\lambda w - w_e)$ is also irreducible.

We put

$$\alpha := \begin{pmatrix} \alpha_2 \\ \vdots \\ \alpha_r \end{pmatrix} \in \mathbb{Z}_{>0}^{r-1}, \quad d := \begin{pmatrix} d_2 \\ \vdots \\ d_r \end{pmatrix} \in \mathbb{Z}_{>0}^{r-1}. \tag{5.3}$$

Let $\sigma_V = (\sigma_{jk})_{1 \leq j, k \leq r}$ be the multiplier matrix of $V$. We note here that by comparison of the degrees in (3.2) the multiplier matrix $\sigma_V$ is described as

$$\sigma_V = \begin{pmatrix} 1 & 0 \\ d - \alpha & \sigma_W \end{pmatrix}, \tag{5.4}$$

where $\sigma_W$ is the multiplier matrix of $W$. Thus $\sigma_W$ is equal to the $(r-1) \times (r-1)$ matrix $\sigma_W = (\sigma_{jk})_{2 \leq j, k \leq r}$.

**Theorem 5.1.** It holds that $\alpha = \sigma_W(1 - e)$ with $e = t(\varepsilon_2, \ldots, \varepsilon_r)$ as above. In other words,

$$\alpha_j = \sum_{k=2}^{r} \sigma_{jk}(1 - \varepsilon_k) \quad (j = 2, \ldots, r). \tag{5.5}$$

Moreover, the multiplier matrix $\sigma_V$ of $V$ is written as

$$\sigma_V = \begin{pmatrix} 1 & 0 \\ \sigma_W e & \sigma_W \end{pmatrix}. \tag{5.6}$$
Proof. We shall prove the theorem by induction on the rank \( r \) of \( V \). We consider the polynomial \( \tilde{P}_j^\xi(\lambda, w_e) \) (\( \lambda \in \mathbb{R} \), \( w_e \in \mathcal{O}_\xi \)) defined in (5.2). First, we assume that \( r = 2 \). In this case, we have \( W = \mathbb{R}c_2 \) and \( E = V_{21} \). For the case \( \varepsilon_2 = 0 \), we have \( \mathcal{O}_\varepsilon = \{0\} \) and hence \( \tilde{P}_2^\xi(\lambda, 0) = \lambda \). If \( \varepsilon_2 = 1 \), then \( \tilde{P}_2^\xi(\lambda, w_e) = \lambda - w_e \) does not have the factor \( \lambda \). Thus in both cases we have \( \alpha_2 = 1 - \varepsilon_2 \). Next we assume that \( r \geq 3 \), and that the theorem is true for clans of rank \( r - 1 \). Let us put \( W' = \bigoplus_{2 < j < k \leq r} V_{kj} \) and \( E' = \bigoplus_{k \geq 2} V_{k2} \). Then

\[
W = \mathbb{R}c_2 + E' \oplus W'.
\] (5.7)

Applying Proposition 2.1 to this decomposition, we see that the following linear map \( \varphi': W' \to \mathcal{L}(E') \) is a representation of \((W', \nabla)\):

\[
\varphi'(w') \xi := \xi \nabla w' \quad (w' \in W', \ \xi \in E').
\]

We write general elements \( w \in W \) as \( w = w_2c_2 + \xi' + w' \) (\( w_2 \in \mathbb{R}, \ \xi' \in E' \), \( w' \in W' \)) without any comments. We put \( e' := e - c_2 \), which is the unit element of \( W' \). Let \( \Delta_W^W(w'), \ldots, \Delta_W^W(w') \) be the basic relative invariants of \( W' \). The equation (5.4) for \( \sigma_W' \) tells us that the multiplier matrix \( \sigma_W' \) of \( W' \) is equal to the \((r - 2) \times (r - 2)\) matrix \( \sigma_W' = (\sigma_{jk})_{3 \leq j, k \leq r} \). We consider the Lie algebra \( \mathfrak{h}' = \{L_w'; \ w' \in W'\} \) of the left multiplication operators of \( W' \) and the corresponding Lie group \( H' = \exp \mathfrak{h}' \). Let \( \Omega' := H'e' \) be the homogeneous cone associated with \((W', \Delta')\). For each \( \delta = (\delta_3, \ldots, \delta_r) \in \{0, 1\}^{r-2} \), we put \( c'_\delta := \delta_3c_3 + \cdots + \delta_rc_r \). Then \( c'_\delta \in \Omega' \) and let \( \mathcal{O}'_\delta := H'c'_\delta \subset \Omega' \). Moreover, for \( j = 3, \ldots, r \), let \( \alpha'_j : \{0, 1\}^{r-2} \to \mathbb{Z}_{\geq 0} \) and polynomials \( P_j^\delta(\lambda, y_\delta) \) (\( \lambda \in \mathbb{R}, \ y_\delta \in \mathcal{O}'_\delta \)) be

\[
\alpha'_j(\delta) := \sum_{k=3}^r \sigma_{jk}(1 - \delta_k), \quad P_j^\delta(\lambda, y_\delta) := \Delta_j^W(\lambda e' - y_\delta) \quad (j = 3, \ldots, r).
\] (5.8)

By the induction hypothesis, there exist irreducible polynomials \( F_j^\delta(\lambda, y_\delta) \) (\( j = 3, \ldots, r \)) such that

\[
P_j^\delta(\lambda, y_\delta) = \lambda^{\alpha'_j(\delta)} F_j^\delta(\lambda, y_\delta) \quad (\lambda \in \mathbb{R}, \ y_\delta \in \mathcal{O}'_\delta).
\] (5.9)

In order to know what power of \( \lambda \) is factored out from \( \tilde{P}_j^\xi(\lambda, w_e) \), it is clearly sufficient by continuity that we argue by restricting the variable \( w_e \) to \( \mathcal{O}_\xi \). Thus we assume \( w_e \in \mathcal{O}_\varepsilon \) and take \( h \in H_W \) such that \( w_e = hc_e \). We note that \( h \) is decomposed as \( h = (\exp T_{22})(\exp \xi'_h)h' \), where \( T_{22} = (2 \log h_{22})L_{c_2} \) with some \( h_{22} > 0 \), \( \xi'_h \in E' \) and \( h' \in H' \). Let \( Q' \) be the symmetric bilinear map associated with \( \varphi' \) and we put

\[
\varepsilon' := \varepsilon(\varphi') = \begin{pmatrix}
\varepsilon_3' \\
\vdots \\
\varepsilon_r'
\end{pmatrix} \in \{0, 1\}^{r-2},
\]

so that \( \varphi' \) is an \( \varepsilon' \)-representation. By Theorem 3.4 applied to \( W \) with the decomposition (5.7) and by the induction hypothesis, we have

\[
\Delta_2^W(w) = w_{22}, \quad \Delta_j^W(w) = (w_{22})^{-\alpha'_j(\varepsilon')} \Delta_j^W(w_{22}w' - \frac{1}{2} Q' [\xi']) \quad (j \geq 3).
\] (5.10)
Let us put \( \hat{e} = \xi(\varepsilon_3, \ldots, \varepsilon_r) \in \{0, 1\}^{r-2} \). Applying Lemma 2.3 to the clan \( W \) with (5.7) for \( y = c_\hat{e} \), we obtain
\[
w_{\hat{e}} = h c_{\hat{e}} = \varepsilon_2(h_{22})^2 c_2 + \varepsilon_2 h_{22} \xi^t + \left( h' c'_{\hat{e}} + \frac{\varepsilon_2}{2} Q'[\xi_{\hat{H}}] \right).
\]
Putting \( y_{\hat{e}} = h' c'_{\hat{e}} \), we have
\[
\lambda c_{\hat{e}} - w_{\hat{e}} = (\lambda - \varepsilon_2(h_{22})^2) c_2 - \varepsilon_2 h_{22} \xi_{\hat{H}} + \left( \lambda c'_{\hat{e}} - y_{\hat{e}} - \frac{\varepsilon_2}{2} Q'[\xi_{\hat{H}}] \right).
\]
(5.11)

(i) The case \( j = 2 \). In this case, we have \( \tilde{P}^{\hat{e}}_\lambda(\lambda, w_{\hat{e}}) = \lambda - \varepsilon_2(h_{22})^2 \). If \( \varepsilon_2 = 0 \) then \( \tilde{P}^{\hat{e}}_\lambda(\lambda, w_{\hat{e}}) = \lambda \), and if \( \varepsilon_2 = 1 \) then \( \tilde{P}^{\hat{e}}_\lambda(\lambda, w_{\hat{e}}) \) does not have the factor \( \lambda \). Hence in both cases we have \( \alpha_2 = 1 - \varepsilon_2 \). Since \( \sigma_{2k} = \delta_{2k} \) \((k = 2, \ldots, r)\), we obtain (5.5) for \( \alpha_2 \).

(ii) The case \( j = 3, \ldots, r \). (a) We first assume that \( \varepsilon_2 = 0 \). In this case, (5.11) reduces to \( \lambda c_{\hat{e}} - w_{\hat{e}} = \lambda c_2 + (\lambda c'_{\hat{e}} - y_{\hat{e}}) \). Let \( d'_{\hat{e}} \) be the homogeneous degree of \( \Delta_{j}^W(w') \) \((j = 3, \ldots, r)\). Using (5.10) and (5.9), we obtain
\[
\Delta_{j}^W(\lambda c_2 + (\lambda c'_{\hat{e}} - y_{\hat{e}})) = \lambda^{a(j)(\hat{e})} + d'_{\hat{e}} \tilde{P}^{\hat{e}}_\lambda(\lambda, y_{\hat{e}})
\]
Here the induction hypothesis for (5.6) says \( \sigma_{j2} = \sum_{k=2}^r \sigma_{jk} \xi'_{\hat{e}} \). By using (1.5) for \( d'_{\hat{e}} \), we can rewrite \( \sigma_{j2} \) as \( -\lambda a(j)(\hat{e}) + d'_{\hat{e}} \). Thus we get
\[
\alpha_j = \sigma_{j2} + a(j)(\hat{e}) = \sum_{k=2}^r \sigma_{jk}(1 - \varepsilon_k) \quad (j = 3, \ldots, r).
\]

(b) Next let us consider the case \( \varepsilon_2 = 1 \). We assume that \( \lambda \) is in a small open neighborhood \( U_0 \) of 0 and \( h_{22} \) in a small open neighborhood \( U_1 \) of 1, so that putting \( a_\lambda := -\lambda - (h_{22})^2 \), we have \( a_\lambda > 0 \). Then by (5.10) and (5.11)
\[
\Delta_{j}^W(\lambda c_{\hat{e}} - w_{\hat{e}}) = (-a_\lambda)^{-a(j)(\hat{e})} \Delta_{j}^W((-a_\lambda)(\lambda c'_{\hat{e}} - y_{\hat{e}} - \frac{1}{2} Q' [\xi_{\hat{H}}]) - \frac{1}{2} Q' [h_{22} \xi_{\hat{H}}])
\]
\[
= (-a_\lambda)^{-a(j)(\hat{e})} \Delta_{j}^W(-a_\lambda \lambda c'_{\hat{e}} + \frac{1}{2} Q' [\xi_{\hat{H}}] + a_\lambda y_{\hat{e}}).
\]
Since \( a_\lambda > 0 \) and \( Q'[\xi_{\hat{H}}] \in \overline{\mathcal{O}_x} \subset \overline{\mathcal{O}} \), we have \( a_\lambda \lambda c'_{\hat{e}} + \frac{1}{2} Q'[\xi_{\hat{H}}] \in \mathcal{O}' \) for any \( \lambda \in U_0 \) and \( h_{22} \in U_1 \). Thus for each such \( \lambda \) and \( h_{22} \), there exists a unique \( g_{\lambda} \in H' \) so that \( g_{\lambda} \lambda c'_{\hat{e}} = a_\lambda \lambda c'_{\hat{e}} + \frac{1}{2} Q'[\xi_{\hat{H}}] \). The one-dimensional representation associated with \( \Delta_{j}^W \) being \( \lambda g_{\lambda} \), we have \( \lambda g_{\lambda}^t(g_{\lambda}) = \lambda_{g_{\lambda}}^W(\lambda \lambda c'_{\hat{e}} + \frac{1}{2} Q'[\xi_{\hat{H}}]) \). Using the relative \( H' \)-invariance of \( \Delta_{j}^W \) and (5.9), we obtain
\[
\Delta_{j}^W(\lambda c_{\hat{e}} - w_{\hat{e}}) = (-a_\lambda)^{-a(j)(\hat{e})} \Delta_{j}^W(-a_\lambda g_{\lambda} \lambda c'_{\hat{e}} + a_\lambda y_{\hat{e}})
\]
\[
= (-a_\lambda)^{-a(j)(\hat{e})} \lambda g_{\lambda}^t(g_{\lambda}) \Delta_{j}^W(-\lambda c'_{\hat{e}} + a_\lambda g_{\lambda}^{-1} y_{\hat{e}})
\]
\[
= (-1)^{a(j)} (-a_\lambda)^{-a(j)(\hat{e})} \Delta_{j}^W((-a_\lambda)(\lambda c'_{\hat{e}} - \frac{1}{2} Q'[\xi_{\hat{H}}]) \Delta_{j}^W(\lambda c'_{\hat{e}} - a_\lambda g_{\lambda}^{-1} y_{\hat{e}})
\]
\[
= \lambda a(j)(\hat{e}) \hat{f}_{\hat{e}}(a_\lambda, \frac{1}{2} Q'[\xi_{\hat{H}}]) \hat{f}_{\hat{e}}(\lambda, a_\lambda g_{\lambda}^{-1} y_{\hat{e}}).
\]
To continue, we introduce a rational function $\tilde{F}_j^\varepsilon(\lambda, w_\varepsilon)$ defined by

$$
\tilde{F}_j^\varepsilon(\lambda, w_\varepsilon) := \lambda^{-\alpha_j(\varepsilon)/2}P_j^\varepsilon(\lambda, w_\varepsilon) = F_j^\varepsilon(-a_\lambda, \frac{1}{2}Q[\varepsilon_h])F_j^\varepsilon(\lambda, a_\lambda g_\lambda^{-1} y_\varepsilon).
$$

We shall show that $\tilde{F}_j^\varepsilon(\lambda, w_\varepsilon)$ is actually an irreducible polynomial. Since $\tilde{P}_j^\varepsilon(\lambda, w_\varepsilon)$ is a polynomial, it is sufficient to prove the existence of a non-zero limit of $\tilde{F}_j^\varepsilon(\lambda, w_\varepsilon)$ as $\lambda \to 0$. Since both of $F_j^\varepsilon$ and $\tilde{F}_j^\varepsilon$ are polynomial functions, and since the map $g: U_0 \ni \lambda \mapsto g_\lambda^{-1} \in H'$ is continuous as well as $\lambda \mapsto g_\lambda$, we obtain

$$
\lim_{\lambda \to 0} \tilde{F}_j^\varepsilon(\lambda, w_\varepsilon) = F_j^\varepsilon(-a_0, \frac{1}{2}Q[\varepsilon_h])F_j^\varepsilon(0, a_0 g_0^{-1} y_\varepsilon).
$$

In order to see that this limit is non-zero, we put $h_{22} = 1$ and $\varepsilon_h = 0$. Then we have $a_0 = 1$ and $g_0 e' = e'$, that is, $g_0 \in H'$ is the identity operator. By (5.9) and (5.8), we have

$$
F_j^\varepsilon(-1, 0) = (-1)^{-\alpha_j(\varepsilon')}P_j^\varepsilon(-1, 0) = (-1)^{d_j - \alpha_j(\varepsilon')}.
$$

On the other hand, since $F_j^\varepsilon(\lambda, y_\varepsilon)$ does not have the factor of $\lambda$, we can take $z_\varepsilon \in \mathcal{O}_\varepsilon$ such that $F_j^\varepsilon(0, z_\varepsilon) \neq 0$. Thus we obtain

$$
\tilde{F}_j^\varepsilon(0, c_2 + z_\varepsilon) = F_j^\varepsilon(-1, 0)\tilde{F}_j^\varepsilon(0, z_\varepsilon) = (-1)^{d_j - \alpha_j(\varepsilon')}\tilde{F}_j^\varepsilon(0, z_\varepsilon) \neq 0.
$$

Hence $\tilde{F}_j^\varepsilon(\lambda, w_\varepsilon)$ does not have the factor of $\lambda$. Since $U_0$ and $U_1$ are open sets and since we now know that $\tilde{F}_j^\varepsilon(\lambda, w_\varepsilon)$ is a polynomial, the function $\tilde{F}_j^\varepsilon(\lambda, w_\varepsilon)$ is extended to $\mathbb{R} \times \mathcal{O}_\varepsilon$ and does not have the factor of $\lambda$. Therefore $\tilde{P}_j^\varepsilon(\lambda, w_\varepsilon) = \lambda^{\alpha_j(\varepsilon)}\tilde{F}_j^\varepsilon(\lambda, w_\varepsilon)$ is an irreducible factorization. This shows

$$
\alpha_j = \alpha_j(\varepsilon') = \sum_{k=2}^r \sigma_{jk}(1 - \varepsilon_k) \quad (j = 3, \ldots, r).
$$

It remains to show (5.6). Since $\alpha = \sigma_W(1 - \varepsilon)$ and $d = \sigma_W 1$ by (1.5) for $\Delta^W_j$, we have $d - \alpha = \sigma_W \varepsilon$. Hence (5.6) follows from (5.4).

**Remark 5.2.** If we put $w_\varepsilon := \sum_{k=2}^r \varepsilon_k \lambda_k c_k$ ($\lambda_k > 0$) in (5.2), then we have

$$
\tilde{P}_j^\varepsilon(\lambda, w_\varepsilon) = \prod_{k=2}^r (\lambda - \varepsilon_k \lambda_k)^{\sigma_{jk}} = \lambda^{\sum_{k=2}^r \sigma_{jk}} \prod_{\varepsilon_k = 0}^{\varepsilon_k} (\lambda - \lambda_k)^{\sigma_{jk}}.
$$

But this only implies that $\alpha_j \leq \sum_{\varepsilon_k = 0}^{\varepsilon_k} \sigma_{jk} = \sum_{k=2}^r \sigma_{jk}(1 - \varepsilon_k)$, since, in general, the restriction to a lower dimensional set of an irreducible polynomial need not be irreducible.

**6. Main theorems**

In this section, we determine the multiplier matrix $\sigma_V$ of $V$. For $k = 1, 2, \ldots, r - 1$, let $V[k]$ and $E[k]$ be the subspaces of $V$ defined respectively by

$$
V[k] := \bigoplus_{k < \ell \leq m \leq r} V_{ml}, \quad E[k] := \bigoplus_{m > k} V_{mk}.
$$
We note here that \( V^{[1]} \) and \( E^{[1]} \) are \( W \) and \( E \) in (2.1) which we have worked with in the previous sections. Then analogously to that situation, we see that \( V^{[k]} \) is a subclan of \((V, \nabla)\) and \( E^{[k]} \ominus V^{[k]} \subset E^{[k]} \). The latter property allows us to define \( \mathcal{R}^{[k]} : V^{[k]} \rightarrow \mathcal{L}(E^{[k]}) \) by

\[
\mathcal{R}^{[k]}(x^k)\xi^k = \xi^k \ominus x^k \quad (x^k \in V^{[k]}, \ \xi^k \in E^{[k]} \text{ and } k = 1, 2, \ldots, r - 1).
\]

Then similarly to Proposition 2.1, each \((\mathcal{R}^{[k]}, E^{[k]})\) is a representation of \((V^{[k]}, \nabla)\). Let \( \varepsilon^{[k]} := \varepsilon(\mathcal{R}^{[k]}) \in \{0, 1\}^{r - k} \), and consider the \( r \times r \) matrix \( \mathcal{E}_k \) defined by

\[
\mathcal{E}_k := \begin{pmatrix}
I_{k-1} & 0 & 0 \\
0 & 1 & 0 \\
0 & \varepsilon^{[k]} & I_{r-k}
\end{pmatrix} \quad (k = 1, \ldots, r - 1).
\]

**Theorem 6.1.** The multiplier matrix \( \sigma_V \) of the clan \( V \) is given by \( \sigma_V = \mathcal{E}_{r-1} \mathcal{E}_{r-2} \cdots \mathcal{E}_1 \).

**Proof.** We shall prove the theorem by induction on the rank \( r \) of \( V \). Let us decompose \( V \) as in (2.1), and let \( \varphi \) be the representation of \((W, \nabla)\) in (2.3). By the induction hypothesis, the multiplier matrix \( \sigma_W \) of \( W \) is described as

\[
\sigma_W = \mathcal{E}_{r-1} \mathcal{E}_{r-2} \cdots \mathcal{E}_2, \quad \mathcal{E}_k = \begin{pmatrix}
I_{k-2} & 0 & 0 \\
0 & 1 & 0 \\
0 & \varepsilon^{[k]} & I_{r-k}
\end{pmatrix} \quad (k = 2, 3, \ldots, r - 1). \quad (6.1)
\]

Let us put \( \varepsilon = \varepsilon(\varphi) \). Applying Theorem 5.1 to \( W \), we have by (6.1)

\[
\sigma_V = \begin{pmatrix}
1 & 0 \\
\sigma_W & \varepsilon
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & \varepsilon & I_{r-1} \\
0 & \varepsilon & I_{r-1} & \mathcal{E}_{r-1} \\
0 & \varepsilon & I_{r-1} & \mathcal{E}_{r-2} \cdots \mathcal{E}_2 \mathcal{E}_1
\end{pmatrix}.
\]

By noting \( \mathcal{E}_k = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \) \( (k = 2, 3, \ldots, r - 1) \), the proof is completed.

We are now able to describe the basic relative invariants \( \Delta^V_1(x), \ldots, \Delta^V_r(x) \) of \( V \). To do so, we introduce the polynomials \( D_j(x) \) \( (j = 1, \ldots, r) \) defined by Vinberg [8, the formula (25), p. 385] (see also Ishi [3, p. 158]). Let \( \| \cdot \| \) be the norm on \( V \) defined by \( \|x\|^2 := (x | x) \) \( (x \in V) \). Given \( x \in V \), we introduce elements

\[
x^{(j)} = \sum_{k=j}^{r} x^{(j)}_{kk} c_k + \sum_{m > k \geq j} X^{(j)}_{mk} \in V^{[j-1]} \quad (j = 1, \ldots, r; \ V^{[0]} := V)
\]

inductively by \( x^{(1)} := x \) and, for \( i = 1, \ldots, r - 1 \),

\[
\begin{aligned}
x^{(i+1)}_{kk} := x^{(i)}_{kk} - \frac{1}{2s_0(c_k)} \|X^{(i)}_{ki}\|^2, \\
x^{(i+1)}_{mk} := x^{(i)}_{mk} X^{(i)}_{mk} - X^{(i)}_{mi} \triangle X^{(i)}_{ki} \quad (i < k < m \leq r).
\end{aligned}
\]

Then the polynomials \( D_j(x) \) are defined by

\[
D_j(x) := x^{(j)}_{jj} \in \mathbb{R} \quad (j = 1, \ldots, r).
\]
The polynomials $D_j(x)$ appear in the solution $h \in H$ of the equation $he_V = x$ for a given $x \in \Omega$. In fact, Vinberg [8, Chapter III, Section 3] tells us that the numbers $h_{jj} > 0$ for $T_{jj}$ in (1.3) are given by

$$h_{11}^2 = D_1(x), \quad h_{jj}^2 = D_1(x)^{-1} \cdots D_{j-1}(x)^{-1}D_j(x) \quad (j = 2, \ldots, r).$$

Using these $h_{jj}$, we have $\Delta_j^V(x) = (h_{11})^{2\sigma_1} \cdots (h_{jj})^{2\sigma_j}$ by definition of $\sigma_{jk}$. From these observations, we obtain the following theorem, which tells us explicitly by what powers of $D_1(x), \ldots, D_{j-1}(x)$ we have to divide $D_j(x)$ in order to obtain the irreducible factor $\Delta_j^V(x)$.

**Theorem 6.2.** The basic relative invariants $\Delta_j^V(x)$ of $V$ are written as

$$\Delta_1^V(x) = D_1(x), \quad \Delta_j^V(x) = \frac{D_j(x)}{\prod_{i<j} D_i(x)^{-\sigma_{ji} + \sigma_{ji+1} + \cdots + \sigma_{jj}}} \quad (j = 2, \ldots, r). \quad (6.2)$$

**Remark 6.3.** Let us verify that the power of $D_i(x)$ in the denominator in (6.2) is non-negative. Indeed, thanks to (5.4), it is sufficient to consider the case $i = 1$. By (5.6), we have $\sigma_{j1} = \sum_{k=2}^j \sigma_{jk} \varepsilon_k$, so that

$$-\sigma_{j1} + \sigma_{j2} + \cdots + \sigma_{jj} = \sum_{k=2}^j \sigma_{jk}(-\varepsilon_k + 1) \geq 0 \quad (j = 2, \ldots, r).$$

**Example 6.4.** (1) Let $\Omega$ be an irreducible symmetric cone of rank $r \geq 3$. The corresponding clan is $V = \text{Herm}(r, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) with the product $\Delta$ defined by $x \Delta y = xy + y(x)^T$. In this case, the polynomials $D_j(x)$ are calculated as

$$D_j(x) = \begin{cases} \det[x], & (j = 1, 2), \\ \det[1] \det[2] \cdots \det[j-2] \det[j-1] \det[j] & (j \geq 3), \end{cases} \quad (6.3)$$

where $\det[k](k = 1, \ldots, r)$ are the left upper corner principal minors of $x$. Since it is clear from a glance at (6.3) that the basic relative invariants $\Delta_j(x)$ of $\Omega$ are equal to $\det[j](x)$ $(j = 1, \ldots, r)$, we can verify the formula (6.2). Put $d = \dim_{\mathbb{K}} \mathbb{K}$. Since we have $\dim V_{kk} = d$ $(k < j)$, a straightforward computation yields $e^{i[k]} = \{1, 0, \ldots, 0\} \in \{0, 1\}^{r-k}$ $(k = 1, \ldots, r - 1)$. Thus the multiplier matrix $\sigma$ of $V$ is given by $\sigma = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$. Then $-\sigma_{ji} + \sigma_{j,i+1} + \cdots + \sigma_{jj} = j - i - 1$, and we have for $j \geq 2$

$$\Delta_j(x) = D_j(x)(D_1(x)^{-2}D_2(x)^{-3} \cdots D_{j-1}(x)^{-1})^{-1} = \det[j]x.$$  

(2) We fix a positive integer $p$. Let $\Omega^0$ be the homogeneous cone defined by

$$\Omega^0 := \left\{ X = \begin{pmatrix} \lambda & \xi \\ \xi & x \end{pmatrix}; \quad \lambda \in \mathbb{K}, \, \xi \in \text{Mat}(r \times p, \mathbb{K}), \, x \in \text{Herm}(r, \mathbb{K}) \right\}.$$

In this case, Theorem 5.1 gives a quicker computation now that we have the previous example. The corresponding clan $V$ has the decomposition (2.1) with

$$E = \text{Mat}(r \times p, \mathbb{K}), \quad W = \text{Herm}(r, \mathbb{K}).$$
The representation \( \varphi \) of (2.3) is given by \( \varphi(x)\xi = x\xi \) \((x \in V, \xi \in E)\), where the product on the right hand side is the ordinary matrix multiplication. We have

\[
\varepsilon(\varphi) = \begin{cases} t(1, \ldots, 1, 0, \ldots, 0) & (1 \leq p < r), \\ t(1, \ldots, 1) & (p \geq r) \end{cases}
\]

by definition, and \( \varphi \) is an \( \varepsilon(\varphi) \)-representation. By the preceding example, we know \( \sigma_W \). Thus Theorem 5.1 yields

\[
\alpha = \sigma_W(1 - \varepsilon(\varphi)) = \begin{cases} t(0, \ldots, 0, 1, 2, \ldots, r - p) & (1 \leq p < r), \\ t(0, \ldots, 0) & (p \geq r) \end{cases}
\]

and hence we obtain the formula for the basic relative invariants of \( V \) by Theorem 3.4. This \( \alpha \) is computed in the previous paper [7] in a direct way. We note that the representation \( \varphi \) is regular if and only if \( p \geq r \).

We conclude this paper by considering the actual factorization formula for \( \text{Det} R_x \) \((x \in V)\). More specifically, we will determine the positive integers \( n_1, \ldots, n_r \) with which we have by Theorem 3.1

\[
\text{Det} R_x = \Delta_1^V(x)^{n_1} \cdots \Delta_r^V(x)^{n_r} \quad (x \in V).
\]

We set \( \underline{n} := (n_1, \ldots, n_r) \) in the form of row vector. We call \( \underline{n} \) the basic index of \( V \), and are now going to express \( \underline{n} \) in terms of \( d_{kj} = \dim V_{kj} \) \((j < k)\).

Considering the degrees in (6.4), we have

\[
\dim V = n_1 \deg \Delta_1^V + \cdots + n_r \deg \Delta_r^V.
\]

Let

\[
m_k := \sum_{l \geq k} \dim V_{lk} \quad (k = 1, \ldots, r),
\]

and we put them also in the form of row vector as \( \underline{m} := (m_1, \ldots, m_r) \). We note that \( m_k = 1 + \dim E^k \) for \( k = 1, 2, \ldots, r - 1 \), and \( m_r = 1 \).

**Theorem 6.5.** One has \( \underline{n} = \underline{m} \sigma_{\underline{V}}^{-1} \).

**Proof.** We shall prove the theorem by induction on the rank \( r \) of \( V \). Let us decompose \( V \) as in (2.1). We denote by \( \Delta_j^W \) \((j = 2, \ldots, r)\) the basic relative invariants of \((W, \Delta)\), and by \( \underline{n}' \) the basic index of \( W \). By (6.5) applied to \( W \), we have \( \dim W = \underline{n}' \underline{d} \), where we recall (5.3) for \( \underline{d} \). Since \( W = \bigoplus_{2 \leq j \leq r} V_{kj} \) is the normal decomposition of \( W \), we have \( m'_k := \sum_{l \geq k} \dim V_{lk} = m_k \) for \( k = 2, \ldots, r \). Let \((\varphi, E)\) be the representation of \((W, \nabla)\) in (2.3) and \( Q \) the symmetric bilinear map associated with \( \varphi \). Now Proposition 3.2 together with (3.2) gives

\[
\text{Det} R_x = \lambda^{1 + \dim E - \dim W} \Delta_2^W \left( \lambda w - \frac{1}{2} Q[\xi] \right)^{n'_2} \cdots \Delta_r^W \left( \lambda w - \frac{1}{2} Q[\xi] \right)^{n'_r} = \Delta_1^V(x)^{1 + \dim E - \underline{n}'(\underline{d} - \underline{\alpha})} \Delta_2^V(x)^{n'_2} \cdots \Delta_r^V(x)^{n'_r}.
\]
This tells us that
\[ n = (1 + \dim E - n'(d - \alpha), n') = (m_1 - n'(d - \alpha), n'). \] (6.7)

Then by the induction hypothesis \( n'\sigma_W = m' \), we obtain by (5.4) and (6.7)
\[ n\sigma_V = (m_1 - n'(d - \alpha), n') \begin{pmatrix} 1 & 0 \\ d - \alpha & \sigma_W \end{pmatrix} = (m_1, m') = m. \]

The proof is now completed. \( \blacksquare \)

**Remark 6.6.** The equation (6.7) tells us that \( n_1 = 1 + \dim E - n'(d - \alpha) \). Let us verify here that we actually have \( n_1 \geq 1 \). We first note that Ishi [2, Lemma 3.3 (ii)] implies that for \( O_\varepsilon = H_W\varepsilon \) we have \( \dim \overline{O_\varepsilon} = m'\varepsilon \). Then the fact that \( Q[E] = \overline{O_\varepsilon} \) together with Theorem 6.5 applied to \( W \) gives
\[ \dim E \geq \dim \overline{O_\varepsilon} = m'\varepsilon = n'\sigma_W\varepsilon = n'(d - \alpha), \]
where the last equality follows from (5.6) and (5.4). Hence we obtain \( n_1 \geq 1 \).

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