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Orbits of Distal Actions on Locally Compact Groups

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Abstract. We discuss properties of orbits of (semi)group actions on locally compact groups G, In particular, we show that if a compactly generated locally compact abelian group acts distally on G then the closure of each of its orbits is a minimal closed invariant set (i.e. the action has [MOC]). We also show that for such an action distality is preserved if we go modulo any closed normal invariant subgroup and hence [MOC] is also preserved. We also show that any semigroup action on G has [MOC] if and only if the corresponding actions on a compact invariant metrizable subgroup K and on the quotient space G/K have [MOC].
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1. Introduction

Let X be a Hausdorff space and let Γ be a (topological) semigroup acting continuously on X by continuous self-maps. The action of Γ on X is said to be *distal* if for any two distinct points $x, y \in X$, the closure of $\{(\gamma(x), \gamma(y)) \mid \gamma \in \Gamma\}$ does not intersect the diagonal $\{(a, a) \mid a \in X\}$. It is said to be *pointwise distal* if for each $\gamma \in \Gamma$, the action of $\{\gamma^n\}_{n\in\mathbb{N}}$ on X is distal. The Γ-action on X is said to have [MOC] (minimal orbit closures) if the closure of every Γ-orbit is a minimal closed Γ-invariant set, i.e. for $x, y \in X$, if $y \in \overline{\Gamma(x)}$ then $\overline{\Gamma(y)} = \overline{\Gamma(x)}$. The notion of distality was introduced by Hilbert (cf. Ellis [7], Moore [13]) and studied by many in different contexts, (see Abels [1]-[2], Furstenberg [8], Raja-Shah [17] and the references cited therein).

Let G be a locally compact (Hausdorff) group and let e denote the identity of G. Let Γ be a semigroup acting continuously on G by endomorphisms. Then the Γ-action on G is distal if and only if $e \notin \overline{\Gamma x}$ for all $x \in G \setminus \{e\}$. Note that if the Γ-action on G has [MOC], then it is distal; for if $e \in \overline{\Gamma x}$, then $\{e\} = \overline{\Gamma e} = \overline{\Gamma x}$ and hence $x = e$. What we are interested in is the converse: If the Γ-action on

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G is distal, does it have [MOC]? The answer is known to be affirmative in any of the following cases: (1) G is compact (2) Γ is compact, (3) G is a connected Lie group and Γ is a subgroup of $\mathrm{Aut}(G)$ (4) Γ is a group and G is discrete, or more generally, all Γ-orbits are closed. If Γ is a group and if Γ' is a closed co-compact normal subgroup, then the Γ-action on G has [MOC] if and only if the Γ'-action on G has [MOC] (cf. [13]); it is easy to see that the same equivalence is true for distality. For a general locally compact group G and a group Γ which acts on G by automorphisms, the answer to the above question is not known. But in case of a certain kind of Γ, we get the following:

Theorem 1.1. *Let G be a locally compact group and let Γ be a compactly generated locally compact abelian group such that Γ acts on G by automorphisms. Then the following are equivalent:*

1. *The Γ-action on G is distal*
2. *The Γ-action on G has* [MOC].

Let us now discuss general actions on compact spaces. For a compact space K, let Γ be a semigroup of continuous bijective self-maps of K. Then Γ is a subsemigroup of $C(K)$, the group of all continuous bijective self-maps on K. Let $[\Gamma]$ be the group generated by Γ in $C(K)$. We know that Γ acts distally on K if and only if $E(\Gamma)$, the closure of Γ in K^K with weak topology, is a group (see [7], Theorem 1 which is for group actions and it can easily be seen that the same proof works for semigroup actions). It is obvious that $E(\Gamma)$ is compact since K^K is so. When $E(\Gamma)$ is a group, we have $E(\Gamma) = E([\Gamma])$; moreover, for any $x \in K$, $\overline{\Gamma(x)} = E(\Gamma)(x) = E([\Gamma])(x)$. In particular if K is a compact group and Γ acts on K by automorphisms and $[\Gamma]$ is as above, then the following are equivalent:

1. The Γ-action on K is distal.
2. The $[\Gamma]$-action on K is distal.
3. The Γ-action on K has [MOC].
4. The $[\Gamma]$-action on K has [MOC].

Moreover, for a closed subgroup H of the compact group K which is Γ-invariant (i.e. $\gamma(H) = H$ for all $\gamma \in \Gamma$), the above equivalence is also true for the actions of Γ and $[\Gamma]$ on K/H. Note that for such an H, the corresponding Γ-action on the homogeneous space $K/H = \{xH \mid x \in K\}$ is canonically defined as $\gamma(xH) = \gamma(x)H$ for all $\gamma \in \Gamma$; it is clearly well-defined.

In [17], it is shown that distality of a semigroup action is preserved by factor actions modulo compact invariant subgroups. We show that a similar result holds for [MOC], (see also Remark 2.2).

Theorem 1.2. *Let G be a locally compact group and let Γ be a subsemigroup of* $\mathrm{Aut}(G)$. *Let K be a compact metrizable Γ-invariant subgroup of G. Then the Γ-action on G has* [MOC] *if and only if Γ-actions on both K and G/K have* [MOC].

The following result is about factor actions modulo closed normal invariant subgroups.

Theorem 1.3. *Let G and Γ be as in Theorem 1.1. Let H be a closed normal Γ-invariant subgroup of G. Then the Γ-action on G has* [MOC] *if and only if Γ-actions on both H and G/H have* [MOC].

We will later show that a similar result holds for distality for a larger class of Γ.

A locally compact group G is said to be *distal* (resp. *pointwise distal*) if the conjugacy action of G on G is distal (resp. pointwise distal). A distal group is obviously pointwise distal. Abelian groups, discrete groups and compact groups are obviously distal. Nilpotent groups, connected groups of polynomial growth are distal (cf. [19]) and p-adic Lie groups of type R and p-adic Lie groups of polynomial growth are pointwise distal (cf. Raja [14] and [15]).

In [17], jointly with C. R. E. Raja, we have shown that any locally compact group is pointwise distal if and only if it has shifted convolution property; i.e. for any probability measure μ on G, whose concentration functions do not converge to zero, there exists $x \in \mathrm{supp}\mu$, the support of μ, such that $\mu^n x^{-n} \to \omega_H$, the Haar measure of some compact group H which is normalised by $\mathrm{supp}\mu$. For a probability measure μ on G, the n-th convolution function of μ is defined as $f_n(\mu, C) = \sup_{g\in G} \mu^n(Cg)$, for any compact subset C of G. We say that the concentration functions of μ do not converge to zero if there exists a compact set C such that $f_n(\mu, C) \not\to 0$ as $n \to \infty$, (see [17] for more details). The following corollary is a consequence of Theorem 6.1 of [17] and Theorem 1.1.

Corollary 1.4. *Let G be a locally compact group. Then the following are equivalent:*

1. *G is pointwise distal.*
2. *G has shifted convolution property.*
3. *For every $g \in G$, the conjugation action of $\{g^n\}_{n\in\mathbb{Z}}$ on G has* [MOC].

A locally compact group G is said to be a *generalised FC^--group* (resp. *FC^--nilpotent*) if G has closed normal subgroups $\{G = G_0, \ldots, G_n = \{e\}\}$ such that $G_{i+1} \subset G_i$ and G_i/G_{i+1} is a compactly generated group with relatively compact conjugacy classes (resp. every orbit of the conjugacy action of G on G_i/G_{i+1} is relatively compact) for all $i = 0, 1, \ldots, n-1$. Any compactly generated abelian group (resp. any polycyclic group) is a generalised FC^--group. Any compactly generated group G has polynomial growth if and only if it is FC^--nilpotent; and it is a generalised FC^--group (cf. [12]). Note that generalised FC^--groups are compactly generated (cf. [12], Proposition 2).

Recall that a subgroup Γ of $\mathrm{Aut}(G)$ is said to be equicontinuous (at e) if and only if there exists a neighbourhood base at e consisting of Γ-invariant neighbourhoods; in case of totally disconnected groups, this is equivalent to the existence of a neighbourhood base at e consisting of compact open Γ-invariant subgroups. If Γ is compact, then it is easy to see that Γ is equicontinuous. If G is a totally disconnected group and if Γ has a polycyclic subgroup of finite index and it acts distally on G, then Γ is equicontinuous (cf. [11], Corollary 2.4). If any group Γ acts on G by automorphisms and its image in $\mathrm{Aut}(G)$ is equicontinuous then we say that the Γ-action on G is equicontinuous.

For a totally disconnected locally compact group G, we have the following:

Proposition 1.5. *Let G be a totally disconnected locally compact group and let Γ be a generalised FC^--group which acts on G by automorphisms. Then the following are equivalent.*

1. *The Γ-action on G is distal.*
2. *The Γ-action on G has* [MOC].
3. *The Γ-action on G is equicontinuous.*

In Section 2, we discuss factor actions modulo compact (resp. closed normal) invariant groups and prove Theorem 1.2, Proposition 1.5 and an analogue of Theorem 1.3 for distal actions of a more general class of groups. In Section 3, we prove the equivalence of distality and [MOC] of certain actions, namely, Theorem 1.1. Note that if Γ acts on G by automorphisms, for convenience, Γ is often equated with its image in $\mathrm{Aut}(G)$, whenever there is no loss of any generality.

2. Orbits of Factor Actions

In this section we discuss [MOC] of factor actions modulo compact invariant groups and modulo closed normal invariant groups. We first show that [MOC] is preserved if we go modulo a compact invariant subgroup by proving Theorem 1.2. Before that we prove a proposition which proves a special case of the theorem in case the compact subgroup is a Lie group.

Proposition 2.1. *Let G be a locally compact group and let Γ be a subsemigroup of* $\mathrm{Aut}(G)$*. Let K and L be compact Γ-invariant subgroups of G such that L is a normal subgroup of K and K/L is a Lie group. Then the Γ-action on G/L has* [MOC] *if and only if Γ-actions on both G/K and K/L have* [MOC].

Proof. **Step 1.** Let G, Γ, K and L be as in the hypothesis. One way implication "only if" is easy to prove. Suppose the Γ-action on G/L has [MOC]. Then clearly the Γ-action on K/L also has [MOC], as K is closed and Γ-invariant. Now we want to show that the Γ-action on G/K has [MOC]. Let $x \in G$ and let $yK \in \overline{\Gamma(xK)}$ in G/K for some $y \in G$. Then $yK \subset \overline{\Gamma(x)K} = \overline{\Gamma(x)}K$ and hence $yk \in \overline{\Gamma(x)}$ for some $k \in K$. In particular, we get that $ykL \subset \overline{\Gamma(x)}L = \overline{\Gamma(x)L}$ as L is compact. Hence $ykL \in \overline{\Gamma(xL)}$ in G/L. Since the Γ-action on G/L has [MOC], we get that $\overline{\Gamma(xL)} = \overline{\Gamma(ykL)}$ and hence $x \in \overline{\Gamma(y)K}$ as $k \in K$, $L \subset K$ and both L and K are Γ-invariant groups. This implies that $xK \in \overline{\Gamma(yK)}$ in G/K and hence the Γ-action on G/K has [MOC]. Note that the condition that K/L is a Lie group is not used in the proof of the "only if" statement.

Step 2. Now we prove the "if" statement. Suppose Γ-actions on both G/K and K/L have [MOC]. This implies that the Γ-action on K/L is distal as K/L is a group. We will first show, using compactness of K, that since the Γ-action on G/K has [MOC], it is distal. This, together with the previous assertion, would imply that the Γ-action on G/L is distal. Let $x, y, a \in G$ be such that $\gamma_d(xK) \to aK$ and $\gamma_d(yK) \to aK$ in G/K for some $\{\gamma_d\} \subset \Gamma$. We need to show that $xK = yK$. Since K is compact, it is easy to show that $\gamma(y^{-1}xK) \to eK$. Since $\{eK\}$ is Γ-invariant in G/K, [MOC] of the Γ-action on G/K implies that $y^{-1}xK = eK$, and hence, $xK = yK$.

For any $g \in G$, let $g' = gL$. The map $g \mapsto g'$ is a continuous proper map from G to G/L. Let $x \in G$ and let $y' \in \overline{\Gamma(x')}$ for some $y \in G$. We want to show that $x' \in \overline{\Gamma(y')}$. Then $yK \in \overline{\Gamma(xK)}$, and as the Γ-action on G/K has [MOC], $xK \in \overline{\Gamma(yK)}$. This implies that $xk \in \overline{\Gamma(y)}$ for some $k \in K$, and hence, $x'k' \in \overline{\Gamma(y')}$. Let $\{\gamma_d\}$ and $\{\beta_d\}$ be nets in Γ such that $\gamma_d(x') \to y'$ and $\beta_d(y') \to x'k'$.

Step 3. Let Γ_0 be the closure of the image of Γ in $\mathrm{Aut}(K/L)$. Suppose Γ_0 is compact. Then Γ_0, being a compact semigroup, is a group. Let β and γ be limit points of images of $\{\beta_d\}$ and $\{\gamma_d\}$ in Γ_0 respectively. Then

$$\gamma_d(x'k') \to y'\gamma(k') \in \overline{\Gamma(y')} \quad \text{and} \quad \beta_d(y'\gamma(k')) \to x'k'\alpha(k') \in \overline{\Gamma(y')},$$

where $\alpha = \beta\gamma \in \mathrm{Aut}(K/L)$. Similarly we get that for

$$k_n = k'\alpha(k') \cdots \alpha^{n-1}(k') \in K/L, \quad x_n = x'k_n \in \overline{\Gamma(y')}, \quad \text{for all } n \in \mathbb{N}.$$

As Γ_0 is a compact group, there exists a sequence $\{n_j\} \subset \mathbb{N}$ such that $\alpha^{n_j} \to I$, the identity of $\mathrm{Aut}(K/L)$. Passing to a subsequence if necessary, we may assume that $k_{n_j} \to c' = cL \in K/L$, for some $c \in K$. Hence $x'c' \in \overline{\Gamma(y')}$. Now as $\alpha^{n_j} \to I$,

$$k_{2n_j} = k_{n_j}\alpha^{n_j}(k_{n_j}) \to (cL)^2 = c^2L.$$

Similarly, for all $m \in \mathbb{N}$,

$$k_{mn_j} = k_{n_j}\alpha^{n_j}(k_{n_j}) \cdots \alpha^{(m-1)n_j}(k_{n_j}) \to c^mL \in K/L$$

and $xc^mL \in \overline{\Gamma(yL)}$. Since K/L is a compact (Lie) group, $e' = eL$ is in the closure of $\{c^mL\}_{m\in\mathbb{N}}$ in K/L and hence $x' \in \overline{\Gamma(y')}$, i.e. $\overline{\Gamma(x')} = \overline{\Gamma(y')}$. Hence the Γ-action on G/L has [MOC].

Since K/L is a Lie group, K/K^0L is finite, and hence, $\mathrm{Aut}(K/K^0L)$ is finite. Arguing as above for K^0L in place of L, we get that the Γ-action on G/K^0L has [MOC] and we may assume that $K = K^0L$, i.e. K/L is connected.

Step 4. Let Z be the subgroup of K such that $L \subset Z$ and Z/L is the center of K/L. Then Z and Z^0L are closed and Γ-invariant. Moreover, K/Z is a connected semisimple Lie group and hence its automorohism group is compact. Therefore arguing as in Step 3 for Z in place of L, we get that the Γ-action on G/Z has

[MOC], and since Z/Z^0L is finite, the Γ-action on G/Z^0L also has [MOC]. Now replacing K by Z^0L, we may assume that K/L is a connected abelian Lie group.

Let $[\Gamma]$ be the group generated by Γ in $\mathrm{Aut}(K/L)$. Then $[\Gamma]$ also acts distally on K/L. By Lemma 2.5 of [2], there exists a finite set of compact (normal) $[\Gamma]$-invariant subgroups $\{K_0, \dots, K_n\}$ in K such that $K = K_0 \supset K_1 \supset \cdots \supset K_n = L$ and the image of $[\Gamma]$ in $\mathrm{Aut}(K_i/K_{i+1})$ is finite for each $i \in \{0, \dots, n-1\}$. Arguing as in Step 3 for K_1 in place of L, we get that the Γ-action on G/K_1 has [MOC]. Since the image of Γ in $\mathrm{Aut}(K_i/K_{i+1})$ is finite, using the above argument repeatedly for K_i/K_{i+1} in place of K/L, we get that the Γ-action on G/K_{i+1} has [MOC], $1 \le i \le n-1$. Since $K_n = L$, the Γ-action on G/L has [MOC]. ∎

Proof of Theorem 1.2. Let G, Γ and K be as in the hypothesis. The "only if" statement follows as in Step 1 of the proof of Proposition 2.1. Now we prove the "if" statement. Suppose that Γ-actions on both G/K and K have [MOC]. Hence Γ-actions on G/K, K and G are distal, (see Step 2 of the proof of Proposition 2.1). Let $\mathcal{K}$ consist of closed (compact) Γ-invariant subgroups C of K such that the Γ-action on G/C has [MOC]. Then $\mathcal{K}$ is nonempty as K belongs to $\mathcal{K}$. We put an order on $\mathcal{K}$ by set inclusion. Let $\mathcal{A} = \{K_d\}$ be a totally ordered subset of $\mathcal{K}$. We show that $K' = \cap K_d \in \mathcal{K}$.

For any $x \in G$ and $y \in \overline{\Gamma(x)}K'$, we show that $\overline{\Gamma(x)}K' = \overline{\Gamma(y)}K'$. We know that $\overline{\Gamma(x)}K_d = \overline{\Gamma(y)}K_d$ for each d. First we show that $\cap_d \overline{\Gamma(x)}K_d = \overline{\Gamma(x)}K'$. One way inclusion is obvious. Let $a \in \cap_d \overline{\Gamma(x)}K_d$. Then $C_d = \overline{\Gamma(x)} \cap aK_d \neq \emptyset$ for all d. Here, $\mathcal{A}' = \{C_d\}$ is a collection of compact sets and intersection of finitely many subsets in $\mathcal{A}'$ is nonempty since $\mathcal{A}$ is totally ordered. Hence $\cap_d C_d$ is nonempty. But

$$\cap_d C_d = \cap_d(\overline{\Gamma(x)} \cap aK_d) = \overline{\Gamma(x)} \cap (\cap_d aK_d) = \overline{\Gamma(x)} \cap aK' \neq \emptyset.$$

Hence $a \in \overline{\Gamma(x)}K'$. Therefore, $\cap_d \overline{\Gamma(x)}K_d = \overline{\Gamma(x)}K'$. Similarly, $\cap_d \overline{\Gamma(y)}K_d = \overline{\Gamma(y)}K'$. This implies that $\overline{\Gamma(x)}K' = \overline{\Gamma(y)}K'$ and hence the Γ-action on G/K' has [MOC], i.e. $K' \in \mathcal{K}$.

By Zorn's Lemma, there exists a minimal element in $\mathcal{K}$, say M. Here, M is a compact Γ-invariant subgroup of K such that the Γ-action on G/M has [MOC] and there is no proper subgroup of M in $\mathcal{K}$. We show that $M = \{e\}$. If possible suppose M is nontrivial. Since $M \subset K$ is compact and metrizable and since the Γ-action on M is distal, it is not ergodic and there exists a (nontrivial) irreducible unitary representation χ of M such that $\chi\Gamma$ is finite upto equivalence classes (cf. [3], Theorem 2.1, see also [16] as the action of the group $[\Gamma]$ generated by Γ is also distal). Let $L = \cap_{\gamma\in\Gamma} \ker(\chi\gamma)$. Then L is a proper closed (compact) normal Γ-invariant subgroup of M and since $\chi\Gamma$ is finite upto equivalence classes, M/L is a (compact) Lie group. Moreover, the Γ-action on M/L is distal (cf. [17], Theorem 3.1) and hence it has [MOC]. By Proposition 2.1, we get that the Γ-action on G/L has [MOC]. Hence $L \in \mathcal{K}$, a contradiction to the minimality of M in $\mathcal{K}$. Hence $M = \{e\}$ and the Γ-action on G has [MOC]. This completes the proof. ∎

Remark 2.2. 1. In Theorem 1.2, if G is first countable then, K is also first countable, and hence, it is metrizable.

2. Theorem 1.2 holds in case Γ is a locally compact σ-compact group, (for e.g. $\Gamma = \mathbb{Z}$) and K is not (necessarily) metrizable. As in this case, the group M as above is not necessarily metrizable. Here, $\Gamma \ltimes M$ is locally compact and σ-compact and hence M has arbitrarily small compact normal Γ-invariant subgroups M_d such that $\cap_d M_d = \{e\}$ and M/M_d is second countable and hence metrizable (cf. [9], Theorem 8.7). Now from Theorem 3.1 of [17], if the Γ-action on M is distal then the corresponding Γ-action on M/M_d is also distal and hence not ergodic and we get a proper closed normal Γ-invariant subgroup (of M/M_d, and hence,) of M, denote it by L again, such that M/L is a Lie group. Now the assertion is obvious from the above proof. Note that any compactly generated locally compact group is σ-compact.

The following corollary follows from Theorem 3.1 in [17], Theorem 1.1 in [2] and Theorem 1.2 above since every connected locally compact group has a unique maximal compact normal (characteristic) subgroup such that the quotient is a connected Lie group.

Corollary 2.3. *Let G be a connected locally compact first countable group. Let Γ be a subgroup of* $\mathrm{Aut}(G)$. *Then the Γ-action on G is distal if and only if it has* [MOC].

We now show that [MOC] is preserved by factors modulo closed normal invariant groups. Before that we prove Proposition 1.5 and a Lemma which will be useful in proving Theorem 2.5 below and also Theorem 1.1.

Proof of Proposition 1.5. Let G be a locally compact totally disconnected group and let Γ be a generalised FC^--group acting on G by automorphisms. Let $\Gamma_0 = \{\gamma \in \Gamma \mid \gamma(x) = x \text{ for all } x \in G\}$. Then Γ_0 is a closed normal subgroup of Γ, Γ/Γ_0 is isomorphic to a subgroup of $\mathrm{Aut}(G)$. Also, Γ/Γ_0 is a generalised FC^--group. It is easy to see that we can replace Γ by Γ/Γ_0 and assume that $\Gamma \subset \mathrm{Aut}(G)$. We prove that $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$.

Suppose Γ acts distally on G. As Γ is totally disconnected, it has a compact open normal subgroup C such that Γ/C has a polycyclic subgroup of finite index (cf. [12]). Since C is compact, by Lemma 2.3 of [11], the Γ-action on G is also equicontinuous, (see also 'Note added in Proof' in [11] for non-metrizable groups). Now G has a neighbourhood base at e consisting of open compact subgroups K_d which are Γ-invariant and $\cap_d K_d = \{e\}$. For each d, since G/K_d is discrete, the Γ-action on G/K_d has [MOC]. Let $x \in G$ and let $y \in \overline{\Gamma(x)}$. Then $\overline{\Gamma(x)}K_d = \Gamma(x)K_d = \Gamma(y)K_d = \overline{\Gamma(y)}K_d$ as K_d is open for all d. $\overline{\Gamma(x)} = \cap_d \Gamma(x)K_d = \cap_d \Gamma(y)K_d = \overline{\Gamma(y)}$. This proves that the Γ-action on G has [MOC]. We know that [MOC] implies distality. ■

Lemma 2.4. *Let G be a locally compact group and let Γ be a group acting on G by automorphisms. Suppose that the Γ-action on G/G^0 is equicontinuous. Then there exist open (resp. compact) Γ-invariant subgroups H_d (resp. K_d) such that $H_d = K_d G^0$, K_d is the maximal compact normal subgroup of H_d, $K_d \cap G^0 = \cap_d K_d$ is the maximal compact normal Γ-invariant subgroup of G^0. In particular, if G^0*

has no nontrivial compact normal subgroup, then K_d is totally disconnected and $H_d = K_d \times G^0$, a direct product, for all d.

Proof. Since the Γ-action on G/G^0 is equicontinuous, there exist open almost connected Γ-invariant subgroups H_d such that $\{H_d/G^0\}$ form a neighbourhood base at the identity in G/G^0 consisting of compact open subgroups.

Choose $H = H_d$ for some fixed d. Since H is almost connected, it is Lie projective, and hence, it has a compact normal subgroup C (say) such that H/C is a Lie group with finitely many connected components. Therefore, H/C, and hence, H has a maximal compact normal subgroup; we denote it by C again. Then C is characteristic in H, and in particular, it is Γ-invariant. Let $H' = CG^0$. Then H'/C is the connected component of the identity in the Lie group H/C. Therefore, H' is an open Γ-invariant subgroup in G and $K = C \cap G^0$ is the maximal compact normal subgroup of G^0. Since H'/G^0 is compact and open in G/G^0, passing to a subnet, we may assume that $H_d \subset H'$ for all d. Let $K_d = C \cap H_d$. Then K_d is a compact normal Γ-invariant subgroup in H_d and $H_d = K_dG^0$ as $G^0 \subset H_d$. Since $K = C \cap G^0 \subset H_d$, $K = K_d \cap G^0$ and K_d is the maximal compact normal subgroup in H_d for every d. Also, since $\cap_d H_d = G^0$, we get that $\cap_d K_d = K$. Moreover, if $K_d \cap G^0 = K$ is trivial, then K_d is totally disconnected and $H_d = K_d \times G^0$, a direct product, as both K_d and G^0 are normal in H_d, for all d. ■

To prove Theorem 1.3, in view of Theorem 1.1, it is enough if we prove the same statement for distal actions. Here, we prove the following for distal actions of a more general class of groups.

Theorem 2.5. *Let G be a locally compact group and let Γ be a generalised FC^--group which acts on G by automorphisms. Let H be a closed normal Γ-invariant subgroup. Then the Γ-action on G is distal if and only if Γ-actions on both H and G/H are distal.*

Proof. Let G, H and Γ be as in the hypothesis. Suppose Γ-actions on G/H and H are distal. Then it is easy to see that the Γ-action on G is distal.

Now we prove the converse. Suppose the Γ-action on G is distal. Then the Γ-action on H is also distal. As in the proof of Theorem 1.5, we may assume that $\Gamma \subset \mathrm{Aut}(G)$. We prove that the Γ-action on G/H is distal. By Theorem 3.3 of [17], the Γ-action on G/G^0 is distal and hence equicontinuous (by Proposition 1.5). By Lemma 2.4, there exists an open Γ-invariant subgroup L in G such that $L = KG^0$, where K is the maximal compact normal Γ-invariant subgroup of L. We know that G/L is discrete, and hence, so is G/HL, where HL is an open Γ-invariant subgroup. Therefore, it is enough to prove that Γ acts distally on HL/H. Since HL/H is isomorphic to $L/(L \cap H)$, without loss of any generality, we may assume that $G = L = KG^0$ and K is the maximal compact normal Γ-invariant subgroup in G. In particular, G/K is a connected Lie group.

Here, HK and $K \cap H$ are closed, normal and Γ-invariant subgroups. By Theorem 3.1 of [17] we know that Γ acts distally on G/K, HK/K and on $K/(K \cap H)$; the latter is isomorphic to HK/H. Hence it is enough to prove that

Γ acts distally on the group G/HK which is isomorphic to $(G/K)/(HK/K)$.

Replacing G by G/K and H by HK/K, we may assume that G is a connected Lie group and H is a closed normal Lie subgroup. Let $\mathcal{G}$ be the Lie algebra of G. Since the Γ-action on G is distal, so is the corresponding action of $\{\mathrm{d}\gamma \mid \gamma \in \Gamma\}$ on $\mathcal{G}$ (cf. [2], Theorem 1.1). Equivalently, the eigenvalues of $\mathrm{d}\gamma$ are of absolute value 1, for all $\gamma \in \Gamma$ (cf. [1], Theorem 1′). Since H is normal and Γ-invariant, the Lie algebra $\mathcal{H}$ of H^0 is a Lie subalgebra which is an ideal invariant under $\mathrm{d}\gamma$, for all $\gamma \in \Gamma$, and the Lie algebra of G/H is isomorphic to $\mathcal{G}/\mathcal{H}$. Then the eigenvalues of $\mathrm{d}\gamma$ on $\mathcal{G}/\mathcal{H}$ are also of absolute value 1 for all $\gamma \in \Gamma$. Hence Γ acts distally on G/H (cf. [1], [2]). This completes the proof. ∎

3. Distality and [MOC]

In this section we show that if Γ is a locally compact, compactly generated abelian (resp. Moore) group acting on a locally compact group by automorphisms, then distality and [MOC] of the Γ-action are equivalent. We first prove a proposition which will be useful in proving Theorem 1.1.

Proposition 3.1. *Let G and Γ be as in Theorem 1.1. Suppose that the Γ-action on G is distal. Given a net $\{\gamma_d\}$ in Γ, let*

$$M = \{g \in G \mid \{\gamma_d(g)\}_d \textit{ is relatively compact}\}.$$

Then M is a closed Γ-invariant subgroup.

Proof. It is obvious that M is a subgroup and it is Γ-invariant since Γ is abelian. Therefore $\overline{M}$ is also a Γ-invariant subgroup. If M is trivial, then $M = \overline{M}$. Suppose M is a nontrivial subgroup of G. Without loss of any generality, we may assume that $G = \overline{M}$, i.e. M is dense in G.

Step 1. By Theorem 3.3 of [17], the Γ-action on G/G^0 is distal. Since Γ is a compactly generated locally compact abelian group, it is a generalised FC^--group. By Proposition 1.5, the Γ-action on G/G^0 has [MOC] and the Γ-action on G/G^0 is equicontinuous. By Lemma 2.4, there exists an open (resp. compact) Γ-invariant subgroup H (resp. K) such that $H = KG^0$, where K is the maximal compact normal subgroup of H. Since H is open and Γ-invariant, it is enough to show that $H \subset M$ and hence, we may assume that $G = H$. Here, since K is a maximal compact normal Γ-invariant subgroup, $K \subset M$ and G/K is a connected Lie group without any nontrivial compact subgroup. Moreover, the Γ-action on G/K is distal (cf. [17], Theorem 3.1). Let $\pi : G \to G/K$ be the natural projection. Since K is compact, $\pi(M) = \{gK \in G/K \mid \{\gamma_d(gK)\}_d$ is relatively compact in $G/K\}$ and M is closed if and only if $\pi(M)$ is closed. Moreover, $\pi(M)$ is dense in G/K. Now, we may replace G by G/K and assume that G is a connected Lie group without any nontrivial compact normal subgroup and $\Gamma \subset \mathrm{Aut}(G)$.

Step 2. Since G has no nontrivial compact central subgroup, $\mathrm{Aut}(G)$ is almost algebraic (as a subgroup of $GL(\mathcal{G})$) (cf. [4]), where $\mathcal{G}$ is the Lie algebra of G. Let Γ' be the smallest almost algebraic subgroup containing Γ in $\mathrm{Aut}(G)$. Here Γ' is a an open subgroup of finite index in the Zariski closure $\tilde{\Gamma}$ of Γ in $GL(\mathcal{G})$, hence Γ' and $\tilde{\Gamma}$ have the same connected component of the identity. It follows from Corollary 2.5 of [1], that the unipotent radical U of $\tilde{\Gamma}$ is a closed co-compact normal subgroup of Γ'. Let $P(G)$ denote the space of all regular Borel probability measures on G with weak* topology. Note that $\mathrm{Aut}(G)$ has a natural action on $P(G)$; namely, for any $\alpha \in \mathrm{Aut}(G)$ and $\mu \in P(G)$, $\alpha(\mu)(B) = \mu(\alpha^{-1}(B))$ for all Borel sets B in G (see [10] for generalities on measures on groups). From Corollary 3.4 of [6], we get that for any measure μ in $P(G)$, the U-orbit of μ, and hence, the Γ'-orbit of μ is closed in $P(G)$, i.e. $\{\alpha(\mu) \mid \alpha \in \Gamma'\}$ is closed in $P(G)$.

Step 3. We now prove that $\{\gamma_d\}_d$ is relatively compact in $\mathrm{Aut}(G)$. Suppose $\{\gamma_d\}_d$ is not relatively compact in $\mathrm{Aut}(G)$. Since $\mathrm{Aut}(G)$ is a Lie group, there exists a divergent sequence $\{\gamma'_n\}$ in the set $\{\gamma_d\}_d$, i.e. $\{\gamma'_n\}$ has no convergent subsequence. We know that $\{\gamma'_n(g)\}$ is relatively compact for all g in a dense subgroup M. There exists a countable subgroup $M_1 \subset M$ which is dense in G. Let $M_1 = \{g_i \mid i \in \mathbb{N}\}$. Passing to a subsequence if necessary, we may assume that $\{\gamma'_n(g_i)\}_n$ converges for all i. Let $x_i \in G$ be such that $\gamma'_n(g_i) \to x_i$, $i \in \mathbb{N}$.

Let $\mu = \sum_{i=1}^{\infty}(1/2^i)\delta_{g_i}$ and let $\lambda = \sum_{i=1}^{\infty}(1/2^i)\delta_{x_i}$, where for any $g \in G$, δ_g denotes the Dirac measure at g. Then $\mu, \lambda \in P(G)$ and it is easy to see that $\{\gamma_n(\mu)\}$ converges to λ. Now from Step 2, there exists $\gamma \in \Gamma'$ such that $\gamma'_n(\mu) \to \gamma(\mu) = \lambda$. Since M_1 is dense in G, the support of μ is whole of G. Therefore the support of $\gamma(\mu)$ is also whole of G. Now by Theorem 1.6 of [5], we get that $\{\gamma'_n\}$ is relatively compact and for any limit point β of it, $\beta(\mu) = \gamma(\mu)$. This implies that $\beta(g) = \gamma(g)$ for all $g \in M_1$ and hence $\beta = \gamma$. Therefore, $\gamma'_n \to \gamma$, i.e. $\{\gamma'_n\}$ is convergent.

This contradicts the above assumption that $\{\gamma'_n\}$ is divergent. Hence we have that $\{\gamma_d\}_d$ is relatively compact in $\mathrm{Aut}(G)$. Therefore, $\{\gamma_d(x)\}_d$ is relatively compact for all $x \in G$ and $G = M$, i.e. M is closed. ■

Remark 3.2. From the above proof it is clear that if G is a connected Lie group without any nontrivial compact central subgroup, Γ is a subgroup of $\mathrm{Aut}(G)$ acting distally on G and if $\{\gamma_d\} \subset \Gamma$ is such that $\{\gamma_d(g)\}_d$ is relatively compact for all g in a dense subgroup of G, then $\{\gamma_d\}$ is relatively compact in $\mathrm{Aut}(G)$.

Proof of Theorem 1.1. Let G be a locally compact group and let Γ be a compactly generated locally compact abelian group. Suppose that the Γ-action on G has [MOC], then we know that the Γ-action on G is distal.

Now suppose that the Γ-action on G is distal. We show that it has [MOC]. Let $x \in G$ and let $y \in \overline{\Gamma(x)}$. We need to show that $x \in \overline{\Gamma(y)}$. We have that $\gamma_d(x) \to y$ for some $\{\gamma_d\} \subset \Gamma$. Let $M = \{g \in G \mid \{\gamma_d(g)\}_d \text{ is relatively compact}\}$. By Proposition 3.1, M is a closed Γ-invariant subgroup and x, and hence, y belongs to M. Without loss of any generality we may assume that $M = G$. In view of Theorem 1.2 and Remark 2.2, we can go modulo the maximal compact normal subgroup of G^0 which is characteristic in G and assume that G^0 is a Lie group without any nontrivial compact normal subgroup. Note that Γ is a generalised FC^--group and the Γ-action on G/G^0 is distal (by Theorem 3.3 of [17]). Hence from Proposition 1.5, we get that the Γ-action on G/G^0 is equicontinuous. Let $H_d = K_d \times G^0$ be open Γ-invariant subgroups, where K_d are totally disconnected compact Γ-invariant subgroups such that $\cap_d K_d = \{e\}$ in G, (see Lemma 2.4). Then passing to a subnet if necessary, we may assume that $\gamma_d(x) = yk_dg_d = yg_dk_d$, where $k_d \in K_d$ and $g_d \in G^0$, $k_d \to e$, $g_d \to e$. In particular, we get that $\gamma_d^{-1}(y) = x\gamma_d^{-1}(k_d^{-1})\gamma_d^{-1}(g_d^{-1})$. We know that $\{\gamma_d|_{G^0}\}$ is relatively compact, (see Remark 3.2). Let γ be a limit point of $\{\gamma_d|_{G^0}\}$ in $\mathrm{Aut}(G^0)$. Then γ^{-1} is a limit point of $\{\gamma_d^{-1}|_{G^0}\}$ in $\mathrm{Aut}(G^0)$. Therefore, passing to a subnet if necessary, we get that

$$\gamma_d^{-1}(g_d^{-1}) \to \gamma^{-1}(e) = e \quad \text{and} \quad \gamma_d^{-1}(y) = xk'_d\gamma_d^{-1}(g_d^{-1}) \to x$$

where $k'_d = \gamma_d^{-1}(k_d^{-1}) \in K_d$ and $k'_d \to e$ as K_d are Γ-invariant and $\cap_d K_d = \{e\}$. In particular, $x \in \overline{\Gamma(y)}$. Since this is true for any $x \in G$ and any $y \in \overline{\Gamma(x)}$, the Γ-action on G has [MOC]. ■

A locally compact group G is said to be a *central* group or a Z-group if $G/Z(G)$ is compact, where $Z(G)$ is the center of G. It is said to be a *Moore* group if all its irreducible unitary representations are finite dimensional. All abelain groups and all compact groups are Z-groups and Z-groups are also Moore groups. A Moore group has a normal subgroup H of finte index such that $\overline{[H,H]}$ is compact (cf. [18]). It is easy to see from this, that any Moore group G is FC^--nilpotent as $G_0 = G$, $G_1 = H$, $G_2 = \overline{[H,H]}$ and $G_3 = \{e\}$. Since G_0/G_1 is finite, and G_1/G_2 is abelian and G_2/G_3 is compact, we have that the conjugacy action of G on G_i/G_{i+1} has relatively compact orbits for all $i = 0, 1, 2$. Hence any compactly generated Moore group has polynomial growth and it is a generalised FC^--group (cf. [12], Theorem 1, Lemma 1).

Corollary 3.3. *Let G be a locally compact group and let Γ be a compactly generated Moore group acting on G by auotomorphisms. Then the Γ-action on G is distal if and only if it has* [MOC].

The proof of the above corollary is essentially the same as that of Theorem 1.1. As Γ is a Moore group, it has a closed normal subgroup Γ_1 of finite index whose commutator group is relatively compact. (cf. [18], Theorem 1). Then by Lemma 4.1 of [13], it is enough to show that the Γ_1-action on G has [MOC]. Without loss of any generality, we may assume that $[\Gamma, \Gamma]$ is relatively compact and hence it is easy to see that the group M defined in the above proof is Γ-invariant. We will not repeat the proof here.

Remark 3.4. From above, it is obvious that Theorem 1.1 holds for any compactly generated locally compact group Γ such that its commutator subgroup is relatively compact. Moreover from Lemma 4.1 in [13] we know that the action of a group Γ on G has [MOC] if the action of any co-compact subgroup of Γ on G has [MOC]. Hence Theorems 1.1 and 1.3 hold for compact extensions of such a group Γ mentioned above, and in particular, for compact extensions of compactly generated abelian, or more generally, of Moore groups.

We conjecture that Theorem 1.1 holds for an action of any generalised FC^--group. It already holds for the action of such a group on totally disconnected groups, compact groups and connected groups.

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